Ph.D. Prelim in Complex Analysis

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1. Let f be analytic in the unit disk $\mathbf{D}$. Use Cauchy's integral formula to establish the power series representation of f in $\mathbf{D}$. Obtain both an integral formula and a derivative formula for the n-th coefficient.

2. Let Ω be a region and let $\mathcal{F} = \{f : f \text{ is analytic in } \Omega \text{ and } |f(z)| \le 1,\ \forall z \in \Omega\}$. Fix $z_0 \in \Omega$ and show that $\exists\, g \in \mathcal{F}$ such that Re $g'(z_0) \ge$ Re $f'(z_0)$, $\forall f \in \mathcal{F}$.

3. Let f be analytic and nonconstant in a region Ω with $\mu = \text{Ref}$ and $v = \text{Imf } f$.

   (a) Show that $|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$.

   (b) Determine all real numbers a and b such that $au^2 + bv^2$ is harmonic in Ω.

4. Let $\Omega = \{z : |z - i| < 1\}$ and $H = \{z : \text{Im } z > 0\}$. Map $H \backslash \overline{\Omega}$ conformally onto Ω.

5. If p is a polynomial, prove that the series $\sum_{n=0}^{\infty} p(n) z^n$ defines a rational function.

   HINT: Note that any linear combination of rational functions is a rational function.

6. (a) Let f be analytic in the unit disk $\mathbf{D}$ with

   $$\lim_{|z| \to 1-} f(z) = 0 \ .$$

   Prove $f \equiv 0$.

   (b) Let g be analytic in $\mathbf{D}$. Prove that the statement

   $$\lim_{|z| \to 1-} g(z) = \infty$$

   is impossible.

7. Let f be meromorphic in $\mathbf{C}$ and bounded outside of some circle. Determine the form of f as completely as possible.

8. Let $\Gamma = \{z : |z| = 1\}$.

   (a) Show that the mapping $z \longmapsto (z+1)^2$ takes Γ onto the cardioid $r = 2(1 + \cos\theta)$. Sketch this cardioid.

   (b) Let $g(w) = \int_\Gamma \dfrac{z(z+1)}{z^2 + 2z - w} dz$ (Γ traversed once counterclockwise). Use the result of part (a) to sketch a domain containing 0 on which g is analytic.

   (c) Determine $g(0)$ and $g'(0)$.