**Ph.D. Preliminary Examination**

**Complex Analysis**

**September 3, 1993**

**Notations.**

(a) $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ is the open unit disk.

(b) $\overline{\Delta} = \{z \in \mathbf{C} : |z| \leq 1\}$ is the closed unit disk.

(c) $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the extended complex plane.

1. Let $f : \Delta \to \Delta$ be analytic. Suppose there exists $z_0 \in \Delta$ with $f(z_0) = z_0$ and $f'(z_0) = 1$. Prove that $f(z) \equiv z$.

2. Consider the elementary functions:

   (1) $\cos : \mathbf{C} \mapsto \mathbf{C}$;

   (2) $\tan : \mathbf{C} \mapsto \overline{\mathbf{C}}$.

   A. Is tan continuous on $\mathbf{C}$?

   B. Determine

   (a) the range of $\cos z$;

   (b) the range of tan;

   (c) $\cos^{-1}\{\frac{5}{4}\}$;

   (d) $\tan^{-1}\{\infty\}$.

3. Let $L$ denote the line segment joining $-i$ and $i$, and let $\Omega = \overline{\mathbf{C}} \backslash L$.

   A. Map $\Omega$ conformally onto $\Delta$.

   B. Deduce that if $f$ is an entire function such that $f(\mathbf{C}) \cap L = \phi$ then $f(z)$ is constant.

4. Let $z_0$ be a simple pole of a function $f(z)$.

   A. Prove

$$\lim_{r \to 0^+} \int_{\gamma_r} f(z)dz = (b-a)i \ \mathrm{Res}[f, z_0] \ ,$$

   where

$$\gamma_r = \{z_0 + re^{it} : a \le t \le b\} \ .$$

   B. Use this result to evaluate $\int_{-\infty}^{\infty} \dfrac{\sin x}{x} dx$.

5. Suppose $f$ is a nonconstant analytic function in $\Delta$. Show that the function

$$M(r) = \max_{|z|=r} |f(z)|, \ 0 < r < 1 \ ,$$

   is strictly increasing.

6. Let $f(z) = \exp(-\dfrac{1+z}{1-z})$.

   A. Show that $f$ is bounded on $\Delta$.

   B. Determine the range of $f$ on $\Delta$.

7. How many zeros has

$$P(z) = 1 + 2z^4 + \frac{7}{10} z^{10}$$

   in $\Delta$? What are the multiplicities of these zeros?

8. Suppose that the functions $f_n$ are holomorphic in a domain $D$, none of the functions $f_n$ has a zero in $D$, and $\{f_n\}$ converges to $f$ uniformly on compact subsets of $D$. Show that either $f$ has no zeros in $D$ or $f \equiv 0$ in $D$.