Complex Analysis Prelim. (Jan. 2010)

In the following, $\mathbb{D}$ stands for the open unit disk, $\mathbb{C}$ stands for the complex plane.

1. Let f be an entire function and let $g(z) = \overline{f(\overline{z})}$. Show that g is entire.

2. Use contour integration to derive the formula

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}, \quad a,\ b > 0.$$

3. Determine the set in $\mathbb{C}$ on which

$$\sum_{n=0}^{\infty} \left(\frac{1-e^z}{1+e^z}\right)^n$$

converges.

4. Let f be a holomorphic function on $\mathbb{D}$.

(a) Compute the Jacobian of the map f (regarding f as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$), and express it in terms of f or f'.

(b) Give a formula for the area of $f(\mathbb{D})$ in terms of the Taylor coefficients of f.

5. Let f be nonconstant, analytic and satisfy $|f(z)| \le M$ on $\mathbb{D}$. Let $f(0) = a_0$. Show that f has no zeros in the set $\{z : \ |z| < |a_0|/M\}$.

6. Let $A = \{z : \ 3/4 < |z| < 1\}$. Let $f_1(z) = \frac{1}{2z-1}$ and $f_2(z) = \frac{1}{2z-3}$. Is it possible to uniformly approximate f_1 or f_2 on A by functions analytic on $\mathbb{D}$? Justify your answers.

7. Determine a linear fractional transformation L that maps the interval $[-1,\ 1]$ onto $\{e^{i\theta} : \ 0 \le \theta \le \pi\}$ and such that $L(-i) = \infty$.

8. (a) State Rouche's Theorem.

(b) Use Rouche's Theorem to prove Hurwitz's Theorem, which states: If, in a region Ω, the functions $\{f_n\}$ are analytic, have no zeros and converge uniformly to f on compact subsets, then either f is the constant 0 or f has no zeros in Ω.

9. (a) Suppose an entire function maps the real line onto the circle $C = \{z : \ |z| = R\}$, $R > 0$. Show that $f(z) \neq 0$ for all $z \in \mathbb{C}$.

(b) Is it true if the real line is replaced by an arbitrary line?

(c) Is it possible for an entire function to map a circle onto a line?