Complex Analysis Prelim. (Jan. 2009)

In the following, $\mathbb{D}$ stands for the open unit disk, $\mathbb{C}$ stands for the complex plane.

Part 1. Do all of the following problems.

1. Show that a complex polynomial of degree $n > 0$ has precisely n zeros in the complex plane.

2. a) Find all solutions of the equation $z^6 + 1 = 0$.

b) Let $g(z) = z^2\overline{z}$. Find all points where g is complex differentiable.

3. Find an explicit conformal map from the region $G = \mathbb{D}\backslash\{0 \leq x < 1\}$ onto the unit disc $\mathbb{D}$.

4. (a) State and prove the Liouville's Theorem.

(b) Let V be the set of entire functions f such that $|f(z)| \leq C|z|^5$ for some constant C (depending probably on f), determine what type of functions are in V and find the dimension of V.

5. Use residue theory to compute

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + sin^2\theta}.$$

Part 2. Do at least two of the following problems.

6. Let $f(z) = u(z) + iv(z)$ be holomorphic in a neighborhood of the closed unit disc $\mathbb{D}$, where u and v are the real and, respectively, the imaginary part of f. Prove the Schwarz formula:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta + iv(0), \quad |z| < 1.$$

7. Find all entire functions f such that $|f(z)| = 1$ when $|z| = 1$.

8. Give as simple as possible a (product) formula for an entire function F which has a zero of order 1 at each point $c_n = \sqrt{n}$, $n = 1, 2, 3, ...$ and no other zero in $\mathbb{C}$.

9. Find an "explicit" series expansion for a meromorphic function f on $\mathbb{C}$ which has a simple pole with residue n at each positive integer $n = 1, 2, 3, ...$, and is holomorphic at all other points. Be sure to prove all relevant convergence statements.

10. Let $V = \{f \in \mathcal{O}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } |a_n| \leq n^2 \text{ for all } n\}$. Prove that there exists $h \in V$, such that $\left|f'(\frac{1}{2})\right| \leq \left|h'(\frac{1}{2})\right|$ for all $f \in V$.