Complex Analysis Prelim. (August, 2008)

PART I: Do all of the following problems.

1. State the following theorems.

(a) Rouché's Theorem.
(b) Riemann Mapping Theorem.
(c) Runge's Theorem.
(d) Mittag-Leffler Theorem.

2. Use residue theory to evaluate the following integrals.

(a)

$$\int_0^{2\pi} \frac{d\theta}{a + bsin\theta}, \quad where\ a > b > 0.$$

(b)

$$\int_0^{\infty} \frac{x^{-a}}{1+x} dx, \quad where\ 0 < a < 1.$$

3. Find a conformal map of the right half-disk $\{Rez > 0,\ |z| < 1\}$ onto the upper half plane.

4. Consider the unit disk $\mathbf{D}$ and an analytic function $f : \mathbf{D} \longrightarrow \mathbf{D}$. If $f(0) = a \neq 0$, show that f does not have a zero in the disk $\{z : |z| < |a|\}$. (Hint: Use Schwarz Lemma.)

PART II: Do at least four of the following problems.

5. If the fractional linear map $f(z) = \frac{az+b}{cz+d}$ maps $\mathbf{D}$ onto $\mathbf{D}$, what can one say about $a,\ b,\ c,\ d$?

6. If $f(z)$ is an entire function such that $|f(z)| \geq 1$ for all z with $|z| \geq \pi$, then show f is a polynomial.

7. Show that a meromorphic function on the extended complex plane $\mathbb{C}^*$ is rational.

8. Let Ω be a bounded domain with piecewise smooth boundary, and g be a smooth function on $\Omega \cup \partial\Omega$. Show that

$$\int_{\partial\Omega} g(z)dz = 2i \int\int_{\Omega} \frac{\partial g}{\partial \bar{z}} dxdy.$$

9. Let f and g be analytic on a domain Ω and continuous on $\Omega \cup \partial\Omega$. If $|f(z) + g(z)| < |f(z)| + |g(z)|$ on $\partial\Omega$, prove that f and g have the same number of zeros in Ω, counting multiplicity.

10. Show that the product $\prod_{n=1}^{\infty} cos(z/n)$ defines an entire function.