COMPLEX ANALYSIS Ph.D. PRELIMINARY EXAM
June 2008

D denotes the open unit disc $\{z \in \mathbb{C} : \ |z| < 1\}$.
Holomorphic functions are also called analytic functions.
Make sure to show all your work!

1. a) Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in D$. Prove that for $0 < r < 1$, the series converges *absolutely* and *uniformly* on $\{ |z| \leq r\}$.

   b) Show that for any positive integer k the power series $\sum_{n=1}^{\infty} n^k z^n$ has radius of convergence 1 and that its limit equals a rational function on D.

2. How many zeroes does $P(z) = 1 + 3\,z^8 - z^{16}$ have in the unit disc D? Determine the multiplicities of these zeroes!

3. Evaluate

   a) $\int_\gamma (z^2 + 3\overline{z})\,dz$, where γ is the upper half of the unit circle from -1 to $+1$.

   b) $\oint_{|z|=4} \frac{1}{\sin z}\,dz$, where the circle is traversed once counterclockwise.

   c)
   $$\int_{-\infty}^{\infty} \frac{\cos(\alpha\,x)dx}{1+x^2}, \text{ where } \alpha \text{ is real.}$$

4. Suppose h is a holomorphic function on D which satisfies $|h\,(z)| \leq \frac{1}{1-|z|}$ for all $z \in D$. Show that $|h'(0)| \leq 4$.

5. Let g be the holomorphic function defined in a neighborhood of i as the branch of $\sqrt{1-z^2}$ which satisfies $g(i) = \sqrt{2}$.

   a) Show that g can be continued analytically along any curve in $G = \mathbb{C} \setminus \{-1,\ 1\}$.

   b) Can g be continued analytically to define a holomorphic function on G ? Why?

   c) Show that the analytic continuation of g leads to a holomorphic function on $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

6. Let $\{f_n(z), n = 1, 2, ...\}$ be a uniformly bounded sequence of holomorphic functions on D (i.e, there is $C < \infty$ such that $|f_n(z)| \leq C$ for all $z \in D$ and n). Suppose there is a point $a \in D$ such that for each $k = 0, 1, 2, ...$ one has $\lim_{n\to\infty} f_n^{(k)}(a) = 0$. ($f_n^{(k)}$ is the kth derivative of f_n.) Show that $f_n \to 0$ uniformly on each compact subset of D.

7. Characterize all holomorphic functions $f(z)$ in D such that $|f(z)| \leq |\cos(1/z)|$ for all $z \in D$.

8. a) Prove that the infinite product

$$P = \prod_{n=1}^{\infty}(1 + \frac{1}{n^2})$$

converges.

b) Prove that value of P defined in a) equals

$$\frac{e^{\pi} - e^{-\pi}}{2\pi}\ .$$

(Hint: You may use an appropriate formula for the sine function.)

9. Find a conformal map f from the strip $S = \{z : |\mathrm{Re}\, z| < \pi\}$ onto the unit disc D which satisfies $f(0) = 0$. (You may leave the answer as a composition of explicit functions.)

10. Let the complex numbers ω_1 and ω_2 be linearly independent over $\mathbb{R}$, and let $L = \{\omega = m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$.

a) Carefully prove that the series

$$F(z) = \frac{1}{z^2} + \sum_{0\neq\omega\in L}[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}]$$

defines a meromorphic function on $\mathbb{C}$, and describe the poles and their principal parts.

b) Prove that the function F defined in a) has periods L, i.e., for any $\omega \in L$ one has

$$F(z + \omega) = F(z) \text{ for all } z \notin L.$$

(Hint: Take derivatives!)