Complex Prelim, January 2008

1. Suppose $|a| < 1$ and $r \in (0, 1)$. Show that the set of complex numbers z satisfying

$$\left| \frac{z-a}{1-\bar{a}\,z} \right| = r$$

is a circle in the complex plane. Find the center and radius of this circle.

2. Suppose $f(z)$ is analytic in $|z| < 1$ and $(1-|z|^2)f(z)$ is bounded there. Use Cauchy's integral formula to show that $(1-|z|^2)^2 f'(z)$ is also bounded in $|z| < 1$.

3. Let Ω be the complex plane with the interval $[0, \infty)$ on the real axis removed. Let $L(z)$ be a branch of the logarithm on Ω with $L(-1) = -\pi$.
   (1) Find $L(e^{\pi i/2})$ and $L(1+i)$.
   (2) Find z and w in Ω such that $L(zw) \neq L(z) + L(w)$.

4. Let $f(z) = 1/[z(z+1)]$. Find the Laurent series of f in each of the following regions.
   (1) $0 < |z| < 1$.
   (2) $|z+1| < 1$.
   (3) $|z+1| > 1$.

5. Show that a bounded meromorphic function on the complex plane is necessarily a constant.

6. Evaluate the integral

$$\int_{|z|=\pi} \left(\frac{1-z^2}{1+z^2} \right)^2 dz.$$

7. How many analytic functions $f(z)$ are there in Ω with the property that $f(z)^2 + 3if(z) + 4$ is identically zero on Ω? Here Ω is the whole complex plane with the two coordinate axes removed. You must justify your answer.

8. Suppose $\{f_n(z)\}$ is a sequence of analytic functions in $|z| < 1$ with $|f_n(z)| \leq 2$ for all n and all $|z| < 1$. If

$$\lim_{n\to\infty} f_n(z) = f(z)$$

pointwise in $|z| < 1$. Show that $f(z)$ is analytic in $|z| < 1$ and

$$\lim_{n\to\infty} f_n'(z) = f'(z)$$

uniformly on every compact subset of $|z| < 1$.

9. If an entire function $f(z)$ satisfies

$$|f(z)| \leq \frac{1+|z|}{1+\sqrt{|z|}}$$

for all z, show that $f = c$, where c is a constant with $|c| \leq 2(\sqrt{2}-1)$.

10. Suppose $u(z)$ is a complex-valued harmonic function in $|z| < 1$ and

$$\lim_{|z|\to 1^-} u(z) = 0.$$

   (1) Give the ϵ-δ definition for the above limit.
   (2) Show that $u(z)$ is identitically zero in $|z| < 1$.