Complex Prelim, January 2006

1. Suppose
$$p(z) = \sum_{n=0}^{N} a_n z^n$$
is a complex polynomial. Show that
$$\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta = \sum_{n=0}^{N} |a_n|^2.$$

2. If u is a harmonic function defined on the complex plane and f is entire, show that $u \circ f$ is harmonic.

3. Construct a conformal mapping from the first quadrant of the complex plane onto the horizontal strip $|y| < 1$.

4. If $f(z)$ is an entire function and its real part is bounded from below, show that f must be constant.

5. Find the Laurent expansion of the function
$$f(z) = \frac{2}{z(z-1)(z-2)}$$
in the annulus $1 < |z| < 2$.

6. Suppose $f(z)$ is entire and $p(z)$ is a polynomial. If $|f(z)| \le |p(z)|$ for all z, show that there exists a constant c such that $f(z) = cp(z)$.

7. Characterize all anaytic functions $f(z)$ in $|z| < 1$ such that $|f(z)| \le |\sin(1/z)|$ for all $0 < |z| < 1$.

8. Suppose each $f_n(z)$ is analytic in the unit disk $|z| < 1$. If $\sum |f_n(z)|$ converges uniformly for $|z| < 1$, show that $\sum |f_n'(z)|$ converges uniformly for $|z| \le r$, where $r \in (0, 1)$.

9. If $f(z)$ is analytic in $|z| < 1$ and $f'(0) \ne 0$, prove the existence of an analytic function $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of the origin.

10. If $f(z)$ is analytic in $|z| < 1$ and $|f(z)| \le 1$ for all $|z| < 1$, show that $(1-|z|^2)|f'(z)| \le 1$ for all $|z| < 1$.