Prelim in Complex Analysis, August 2003

Let C be the complex plane and $D = \{z \in C : |z| < 1\}$ be the open unit disk.

1. Evaluate the following integrals.

$$\int_{|z|=2} \csc z \, dz, \qquad \int_{|z|=1} \frac{1-\cos z}{z^2}\, dz, \qquad \int_0^{\pi} \frac{d\theta}{2+\cos\theta}.$$

2. If $f(z)$ is entire and $|f(z)| \leq |z|^{3/2}$ for all z, show that f is identically zero.

3. Show that a function $f : D \to C$ is constant if and only if both f and $\overline{f}$ are analytic in D.

4. Show that the class X of analytic functions f in D with

$$\int_D |f(z)|\, dx\, dy \leq 1$$

is a normal family.

5. Find the real and imaginary parts of the complex number

$$z = \text{Log}\,(1+i) + \cos(1+i),$$

where Log is the branch of the logarithm on $C - \{x : x \leq 0\}$ with $\text{Log}\,(1) = 2\pi i$.

6. If $f : D \to C$ is a bounded analytic function, show that

$$\sup_{z\in D}(1-|z|^2)|f'(z)| \leq \sup_{z \in D}|f(z)|.$$

7. Show that if $f : D \to C$ is analytic and one-to-one, then $f'(z) \neq 0$ for every $z \in D$.

8. If $f : D \to D$ is analytic and $f(0) = f'(0) = 0$, show that $|f(z)| \leq |z|^2$ for all $z \in D$.

9. Find the Laurent series of $f(z) = 1/[z(1-z)]$ at $z = 0$, at $z = 1$, at $z = 2$, and at $z = \infty$.

10. Let

$$B(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a}_k z},$$

where $a_1, \cdots, a_n$ are distinct points in $D - \{0\}$. Show that

$$B(z) = \prod_{k=1}^{n} \frac{1}{\overline{a}_k} + \sum_{k=1}^{n} \frac{1}{\overline{a}_k \overline{B'(a_k)}(1 - \overline{a}_k z)}.$$