Ph.D. Preliminary Examination in Algebra

June 9, 2000

1. Let X be any set, F any field, and F^X the set of maps from X to F. F^X is endowed with the structure of a vector space over F using "pointwise" addition and multiplication by scalars. Prove that a finite sequence $f_1, f_2, \ldots, f_n$ of elements of F^X is linearly independent if and only if there is a finite sequence of elements $x_1, x_2, \ldots, x_n$ in X for which the $n \times n$ determinant $\det(f_i(x_j))$ is non-zero.

2. Find a complete set of representatives for the isomorphism classes of finite abelian groups of order 1001.

3. Let K be the splitting field over the field Q of rational numbers of the polynomial
$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1 \quad .$$
   (a) What are the possible values for the **minimum** degree among the irreducible factors of a polynomial of degree 5?
   (b) Write f as the product of factors irreducible over R.
   (c) Write f as the product of factors irreducible over Q.
   (d) What is the degree of K over Q?
   (e) What is the Galois group of K over Q?

4. Show that if two square matrices of the same finite size over a field are similar in a larger field then they must be similar in the original field.

5. Let F be a field, and let A be the quotient ring
$$A = F[t, x, y, z]/(tz - xy)F[t, x, y, z]$$
where t, x, y, z are independent transcendentals over F.
   (a) Show that A has no zero divisors.
   (b) Explain briefly why A is Noetherian.
   (c) Is A a unique factorization domain? (Either prove that it is or exhibit an example of something that does not factor uniquely according to the usual criteria for such uniqueness.)

6. Let E be a finite extension of a field F.
   (a) Outline an argument for showing that if F is a finite field, then E is a cyclic Galois extension of F.
   (b) Provide an example where F is a field of characteristic 5 and E is an extension of F of degree 5 that is not a Galois extension of F.
   (c) For any given field K explain how to obtain an extension F of K and a finite extension E of F for which E is a Galois extension of F with Galois group isomorphic to the symmetric group S_n (consisting of the permutations of n objects).

7. Let $\mathbf{F}_3$ denote the field of 3 elements.
   (a) What is the cardinality of 2-dimensional Cartesian space $\mathbf{F}_3 \times \mathbf{F}_3$ over $\mathbf{F}_3$?
   (b) Let N denote cardinality of the group $\mathrm{GL}_2(\mathbf{F}_3)$ of linear automorphisms of $\mathbf{F}_3 \times \mathbf{F}_3$. Compute N.
   (c) Observe that the multiplicative group $\mathbf{F}_3^*$ is the unique group of order 2 and furthermore that:
      i. Multiplication by invertible scalars gives rise to a homomorphism ϕ from $\mathbf{F}_3^*$ to $\mathrm{GL}_2(\mathbf{F}_3)$.
      ii. The determinant gives rise to a homomorphism ψ from $\mathrm{GL}_2(\mathbf{F}_3)$ to $\mathbf{F}_3^*$

      Explain why the kernel of ψ and the cokernel of ϕ both have the same cardinality.
   (d) Is the kernel of ψ isomorphic to the cokernel of ϕ?

8. Prove over *any commutative ring* (with 1) that two isomorphic free modules of finite rank must have the same rank.