1. Let $A$ be an $n \times n$ matrix with entries in the field $\mathbb{C}$ of complex numbers that satisfies the relation $A^2 = A$. Show that $A$ is similar to a diagonal matrix which has only 0’s and 1’s along the diagonal.

2. Furnish examples of the following:
   (a) A finite group that is solvable but not abelian.
   (b) A finite group whose center is a proper subgroup of order 2.
   (c) A nested sequence of finite groups $G, H, K$ with $H$ a normal subgroup of $G$ and $K$ a normal subgroup of $H$ such that $K$ is not a normal subgroup of $G$.

3. Let $p$ be the polynomial $p(t) = t^5 + t^2 + 1$ regarded as an element of the ring $A = \mathbb{F}_2[t]$ of polynomials with coefficients in the field $\mathbb{F}_2$ of two elements. Show that $p$ is irreducible, and then find a polynomial of degree at most 4 with the property that its residue class modulo the ideal $pA$ generates the entire multiplicative group of units in the quotient ring $A/pA$.

4. Let $G$ be a finite group of order $N$, and let $n$ be a positive integer that divides $N$. Do one of the following:
   (a) Prove that if $G$ is abelian, then $G$ contains a subgroup of order $n$.
   (b) Find an example of $G$, $N$, $n$ as above where $G$ has no subgroup of order $n$.

5. Show that every group of order 30 contains a normal cyclic subgroup of order 15.

6. Let $F$ be the field $\mathbb{Q}(i)$ where $i = \sqrt{-1} \in \mathbb{C}$, and let $E$ be the splitting field over $F$ of the polynomial $f(t) = t^4 - 5$. Find:
   (a) the extension degree $[E : F]$.
   (b) the group $\text{Aut}_F(E)$ of all automorphisms of $E$ that fix $F$.

7. Let $\mathbb{F}_2$ be the field of 2 elements, and let $R$ be the commutative ring
   $$R = \mathbb{F}_2[t]/t^3\mathbb{F}_2[t].$$
   (a) How many elements does $R$ contain?
   (b) What is the characteristic of $R$?
   (c) Find all ring homomorphisms $R \rightarrow R$.

8. Let $a, b, c, d$ be elements of a field $F$, let $A, B, C, D$ be $n \times n$ matrices over $F$, and let
   $$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 
   If $\lambda : F^2 \rightarrow F^2$ and $\Lambda : F^{2n} \rightarrow F^{2n}$ denote the linear endomorphisms corresponding (relative to standard coordinates) to $m$ and $M$, respectively, then to what linear endomorphism that may be constructed from $\lambda$ and $\Lambda$ may one relate the $4n \times 4n$ (Kronecker product) matrix
   $$\begin{pmatrix} aA & bA & aB & bB \\ cA & dA & cB & dB \\ aC & bC & aD & bD \\ cC & dC & cD & dD \end{pmatrix}.$$ 
   Explain your answer.