Ph.D. Preliminary Examination in Algebra

January 25, 1999

1. Let G be a finite group and $\mathrm{Perm}(G)$ the group of permutations of G viewed as a set.

   (a) Show that the map
   $$\lambda : \; G \longrightarrow \mathrm{Perm}(G)$$
   that is defined by $\lambda(\sigma)(\tau) \; = \; \sigma \circ \tau$ is a group homomorphism.

   (b) Show that the map ρ_1 defined by $\rho_1(\sigma)(\tau) \; = \; \tau \circ \sigma$ is a homomorphism if and only if G is an abelian group.

   (c) Show that the map ρ defined by $\rho(\sigma)(\tau) \; = \; \tau \circ \sigma^{-1}$ is a homomorphism for every group G.

2. Let A be an $n \times n$ matrix in a field K, let $c(t)$ be the *characteristic polynomial* of A, and let $m(t)$ be the *minimal polynomial* of A. Show that $m(t)$ divides $c(t)$ in the polynomial ring $K[t]$.

3. Show that the alternating group A_4 has no subgroup of index 2.

4. Let $f(x) \; = \; x^5 - 2$ in $\mathbf{Q}[x]$, and let K be the splitting field of $f(x)$ over $\mathbf{Q}$.

   (a) Find generators for K as a $\mathbf{Q}$-algebra.

   (b) Find the Galois group G of K over $\mathbf{Q}$.

   (c) For each subgroup H of G describe the subfield of K which corresponds to H under the "fundamental correspondence of Galois theory".

5. Show that if a finite ring R admits an injective (ring) homomorphism from a field, then the number of elements of R must be a power of a prime number.

6. Let R be a commutative ring, H a commutative R-algebra, and I an ideal in H. Show that
   $$H/I \otimes_R H/I \;\cong\; \frac{H \otimes_R H}{I \otimes_R H \; + \; H \otimes_R I} \quad .$$

7. Let the field L be a (finite) Galois extension of the field K. Define $\mathrm{tr} : \; L \rightarrow K$ by
   $$\mathrm{tr}(\alpha) \quad = \quad \sum_{\sigma \in G} \sigma(\alpha) \quad .$$
   Show that this *trace* map is surjective on K.

8. Let R be a ring and P a left R-module. Show that the following two statements are equivalent:

   (a) P is a direct summand of a finitely-generated free left R-module.

   (b) There exist $x_1, \ldots, x_n \in P$, and $f_1, \ldots, f_n \in \mathrm{Hom}_R(P, R)$ such that the relation
   $$x \quad = \quad \sum f_i(x) x_i$$
   holds for all $x \in P$.