Topics in Algebraic Geometry (Math 825)

Introduction to Schemes

Outline with Comments

Spring Semester, 2006

Note: If you found this document through a web search engine, you may not be aware of its other presentation formats[1].

1 Outline

Fri., May. 5:

A 1949 paper by André Weil gave evidence for the existence of "topological cohomology" in algebraic geometry linked to the notion of zeta function for a non-singular projective algebraic variety defined over a finite field.

Let X be a scheme of finite type over $\mathbf{Z}$. For each element $x \in X$ the residue field at x is the fraction field of an algebra of finite type over $\mathbf{Z}$. Thus, the residue field at a **closed** element x is a field that is an algebra of finite type over $\mathbf{Z}$, i.e., a finite field. One defines the *zeta function* of X by

$$\zeta_X(s) \;=\; \prod_{x \text{ closed in } X} \frac{1}{1 - N(x)^{-s}}$$

where $N(x) = |\kappa(x)|$ is the number of elements of the residue field of X at x. (Ignore questions of convergence for now.) When $X = \mathrm{Spec}\mathbf{Z}$, $\zeta_X(s)$ is Riemann's zeta function. When X is a scheme of finite type over $\mathbf{F}_q$, each residue field at a closed element is a finite extension field of $\mathbf{F}_q$, and, therefore, $N(x) = q^{d(x)}$ where $d(x)$ is the extension degree. With $t = q^{-s}$ one writes

$$\zeta_X(s) \;=\; Z_X(t) \;=\; \prod_{x \text{ closed}} \frac{1}{1 - t^{d(x)}}$$

With the condition $Z_X(0) = 1$ the Z form of the zeta function is determined by its logarithmic derivative

$$\begin{aligned}
\frac{d}{dt}\log Z_X(t) &= \sum_{x \text{ closed}} d(x)\frac{t^{d(x)-1}}{1 - t^{d(x)}} \\
&= \frac{1}{t}\sum_{r\geq 1}\sum_{\{\,x \text{ closed}\,|\,d(x)\,=\,r\}} r\frac{t^r}{1-t^r} \\
&= \frac{1}{t}\sum_{r\geq 1} rc_r\frac{t^r}{1-t^r} \\
&= \frac{1}{t}\sum_{r\geq 1} rc_r\sum_{s\geq 1} t^{rs} \\
&= \frac{1}{t}\sum_{\nu\geq 1}\sum_{r \text{ divides } \nu} rc_r t^{\nu} \\
&= \sum_{\nu\geq 1} N_\nu t^{\nu-1}
\end{aligned}$$

where c_r denotes the number of closed elements in X with $d(x) = r$ and N_ν denotes the number of points of X with values in the degree ν extension of $\mathbf{F}_q$.

[1]URI: http://math.albany.edu/math/pers/hammond/course/mat825s2006/

For a beginning example, when $X = \mathbf{A}^n$, one has $N_\nu = q^{n\nu}$, and, therefore,

$$Z_{\mathbf{A}^n}(t) = \frac{1}{1-q^n t} \quad .$$

Of course, $\mathbf{A}^n$ is not a projective variety for $n \geq 1$.

When F is a field, the set of F-valued points of $\mathbf{P}^n$ is the disjoint union of $\mathbf{A}^0(F), \mathbf{A}^1(F), \ldots, \mathbf{A}^n(F)$. Therefore, $\mathrm{Dlog} Z_{\mathbf{P}^n}(t)$ (over $\mathbf{F}_q$) is the sum of $\mathrm{Dlog} Z_{\mathbf{A}^j}(t)$ for $0 \leq j \leq n$. Hence,

$$Z_{\mathbf{P}^n}(t) = \frac{1}{(1-t)(1-qt)\ldots(1-q^n t)} \quad .$$

For $\mathbf{P}^1 \times \mathbf{P}^1$, one has $N_\nu = (1+q^\nu)^2$, and, therefore

$$Z_{(\mathbf{P}^1 \times \mathbf{P}^1)}(t) = \frac{1}{(1-t)(1-qt)^2(1-q^2t)} \quad .$$

For curves of genus 1 defined over finite fields, the shape of its Z function was established before the time of Weil's conjectures. For example, in the case of the curve E given by the Weierstrass equation $y^2 = x^3 - 2x$ over the field $\mathbf{F}_5$, simply by counting points to see that $|E(\mathbf{F}_5)| = 10$, it is a consequence of the theoretical framework that $Z(t)$ is the rational function

$$Z_E(t) = \frac{1+4t+5t^2}{(1-t)(1-5t)} \quad .$$

For each of these last examples P^n, $P^1 \times P^1$, and E one may observe that $Z_X(t)$, relative to the field $\mathbf{F}_q$ is a rational function in one variable and that:

1. the denominator is the product of polynomials whose degrees are the classical topological Betti numbers of the base extension $X_{\mathbf{C}}$ of X for even dimensions.
2. the numerator is the product of polynomials whose degrees are the classical topological Betti numbers of the base extension $X_{\mathbf{C}}$ of X for odd dimensions.
3. the polynomial factor corresponding to classical cohomology in dimension j has the form of the characteristic polynomial of a linear endomorphism φ of the form $\det(1 - t\varphi)$ with complex reciprocal roots all of absolute value $q^{j/2}$.

Wed., May. 3:

Beyond the theory of curves of genus 1 a good bit of what is involved in the study of curves and of complete non-singular varieties in general is studying the group $\mathrm{Div}(X)/\mathrm{Div}_\ell(X)$. For curves one has

$$\mathrm{Div}_\ell(X) \subseteq \mathrm{Div}_0(X) \subseteq \mathrm{Div}(X)$$

where the quotient for the second step is the discrete group $\mathbf{Z}$ when $\mathrm{Div}_0(X)$ is defined as the group of divisors of degree 0. It turns out that the quotient for the first step is a complete irreducible group variety of dimension g, and, thus, one cannot study curves in depth without studying varieties of higher dimension.

For varieties of dimension greater than 1, defining the degree of a divisor as the sum of its coefficients will not lead in the right direction. One would like a definition of $\mathrm{Div}_0(X)$ such that the first step is a complete irreducible variety and the second step a finitely-generated abelian group, but there is no hope with these two conditions that the second step will always be cyclic since for the case $X = \mathbf{P}^1_k \times \mathbf{P}^1_k$ one will find that $\mathrm{Div}(X)/\mathrm{Div}_\ell(X) \cong \mathbf{Z} \times \mathbf{Z}$.

For the purpose of gaining insight about $\mathrm{Div}(X)/\mathrm{Div}_\ell(X)$ in the theory of curves while at the same time beginning to understand what might be required for defining $\mathrm{Div}_0(X)$ when $\dim(X) > 1$, consider what is available with transcendental methods when $k = \mathbf{C}$. Complex exponentiation provides the short exact sequence of abelian sheaves for the classical (locally Euclidean) topology on X:

$$0 \to \mathbf{Z} \to \mathcal{O}_{\mathrm{hol}} \xrightarrow{e} \mathcal{O}^*_{\mathrm{hol}} \to 0$$

where $e(f) = e^{2\pi i f}$. In the long cohomology sequence the 0 stage splits off since $H^0(X, \mathcal{O}_{\text{hol}}) \cong \mathbf{C}$. GAGA tells us that coherent module cohomology matches, and although $\mathcal{O}^*$ is certainly not an $\mathcal{O}$-module, its H^1 in both algebraic and transcendental theories viewed through Czech theory classifies isomorphism classes of invertible coherent modules. One has the exact sequence:

$$0 \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_{\text{hol}}) \to H^1(X, \mathcal{O}^*_{\text{hol}}) \to H^2(X, \mathbf{Z}) \quad .$$

If $\dim(X) = 1$, then $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$, and one finds that the last map in this sequence, a "connecting homomorphism", sends the isomorphism class of an invertible $\mathcal{O}_{\text{hol}}$-module to its degree. Therefore, remembering that $\text{Div}(X)/\text{Div}_\ell(X) \cong H^1(X, \mathcal{O}^*)$, one has

$$H^1(X, \mathcal{O}_{\text{hol}})/H^1(X, \mathbf{Z}) \cong \text{Div}_0(X)/\text{Div}_\ell(X) \ ,$$

and, in fact, the left side is the quotient of a g-dimensional vector space over $\mathbf{C}$ by a lattice. Thus, $\text{Div}_0(X)/\text{Div}_\ell(X)$ is a g-dimensional complex torus; it is, moreover, a complete group variety over $\mathbf{C}$.

For $\dim(X) > 1$ the kernel of the connecting homomorphism will provide a correct notion of "degree 0".

For working over an arbitrary algebraically closed field, one sees that something is needed to replace classical cohomology. Because constant sheaves are flasque in the Zariski topology, their Zariski-based cohomology cannot be used.

Mon., May. 1:

Continuing with the discussion of the previous hour: If p, q, r are any three points of $X(k)$, then the triple sum $p + q + r$, like any point of $X(k)$ is characterized by the linear equivalence class of the associated one point divisor. One has the relation of linear equivalence

$$\langle p + q + r \rangle \equiv \langle p \rangle + \langle q \rangle + \langle r \rangle - 2 \langle o \rangle \quad .$$

Therefore,

$$\begin{aligned} p + q + r \ = \ o \Leftrightarrow & \ \langle p \rangle + \langle q \rangle + \langle r \rangle \equiv 3 \langle o \rangle \\ \Leftrightarrow & \ \langle p \rangle + \langle q \rangle + \langle r \rangle \ = \ \text{div}(h) + 3 \langle o \rangle \ \text{ for some } h \in L(3 \langle o \rangle) \\ \Leftrightarrow & \ \langle p \rangle + \langle q \rangle + \langle r \rangle \ = \ \text{div}(s) \text{ for some } s \in H^0(X, \mathcal{O}(3 \langle o \rangle)) \\ \Leftrightarrow & \ \langle p \rangle + \langle q \rangle + \langle r \rangle \ = \ \text{div}(axu^3 + byu^3 + cu^3), \text{ some } (a : b : c) \in \check{\mathbf{P}}^2_k \\ \Leftrightarrow & \ \langle p \rangle + \langle q \rangle + \langle r \rangle \ = \ f^{-1}(D), \ D \ = \ \text{div}(aX + bY + cZ) \in \text{Div}(\mathbf{P}^2_k) \end{aligned}$$

where $f : X \to \mathbf{P}^2_k$ is the projective embedding of X given by the invertible $\mathcal{O}$-module $\mathcal{O}(3 \langle o \rangle)$. In other words, taking multiplicities into consideration, three points sum to $\langle o \rangle$ in the group law on $X(k)$ if and only if the corresponding points of a Weierstrass model in $\mathbf{P}^2_k$, with o corresponding to the point on the line at infinity, are collinear.

From this description of the group law on $X(k)$, in view of the fact that the third point of a cubic on the line through two given points (tangent if the two points coincide) depends rationally on the coordinates of the given points, it follows that

1. Addition $X \times X \to X$ and negation $X \to X$ are morphisms of varieties over k.
2. If F is the field generated over the prime field by the coefficients $a_0, \dots, a_6$ of the Weierstrass equation, then
   (a) The Weierstrass equation defines a scheme X_F of finite type over F whose base extension to k is X.
   (b) For each extension E of F the set $X_F(E)$ is a group in a functorial way.
   (c) $X_F(k) \cong X(k)$.

Fri., Apr. 28:

Continuing with curves of genus 1, we wish to change notation so that the projective embedding of the previous hour is given by the very ample invertible sheaf $\mathcal{O}(3 \langle o \rangle)$, $o \in$

$X(k)$. This notational change notwithstanding, o is an arbitrary point. Under the projective embedding given by $\mathcal{O}(3\langle o\rangle)$, one has $f(o) = (0:0:1)$, the unique point of $f(X)$ on the line at infinity. We wish to show that there is a unique commutative group law on the set $X(k)$ for which the map $\varphi : Div(X) \to X(k)$

$$D \;=\; \sum_{p\in X(k)} n_p \langle p\rangle \longmapsto \varphi(D) \;=\; \sum_{p\in X(k)} n_p p\ ,$$

which is tautologically a group homomorphism, has the property that $\varphi(D_1) = \varphi(D_2)$ whenever $D_1 \equiv D_2$ (linear equivalence), and further the property that o is the zero element in $X(k)$. (This is not the strongest statement of this type that can be made.) Addition in $X(k)$ is defined by observing that since for given $p, q \in X(k)$ the divisor $\langle p\rangle + \langle q\rangle - \langle o\rangle$ has degree 1, its complete linear system consists of a single non-negative divisor of degree 1, i.e., $\langle r\rangle$, and this unique $r \in X(k)$ is defined to be $p + q$. Since

$$\langle p\rangle + \langle q\rangle - \langle o\rangle \equiv \langle r\rangle\ ,$$

the properties specified for φ make this definition necessary if, indeed, it defines a group.

It is straightforward to verify that the addition is associative, that o is its identity, and that $-p$ is given by the unique member of the complete linear system $|2\langle o\rangle - \langle p\rangle|$. It is obvious that this group law on $X(k)$ is commutative and that φ is surjective. If $\mathrm{Div}_0(X)$ denotes the group of divisors of degree 0, then since $\varphi(D) = \varphi(D - (\mathrm{deg}D)\langle o\rangle)$, one sees that the restriction φ_0 of φ to $\mathrm{Div}_0(X)$ is a surjective homomorphism. Let $\mathrm{Div}_\ell(X)$ denote the group of divisors linearly equivalent to zero. It is trivial that the map $D \mapsto D - (\mathrm{deg}D)\langle o\rangle$ defines a homomorphism $\mathrm{Div}(X) \to \mathrm{Div}_0(X)$ which, when followed with reduction provides a homomorphism $\mathrm{Div}(X) \to \mathrm{Div}_0(X)/\mathrm{Div}_\ell(X)$. It is not difficult to verify that another homomorphism between this latter pair of groups is given by

$$D \mapsto \langle\varphi(D)\rangle - \langle o\rangle \bmod \mathrm{Div}_\ell(X)\quad .$$

(That this is a homomorphism follows from reviewing the definition of $\varphi(D_1) + \varphi(D_2)$.) Since these two homomorphisms agree on divisors of the form $\langle p\rangle$ – which generate the free abelian group $\mathrm{Div}(X)$ –, one has for all $D \in \mathrm{Div}(X)$ that

$$D - (\mathrm{deg}D)\langle 0\rangle \equiv \langle\varphi(D)\rangle - \langle o\rangle\quad .$$

We know that $\mathrm{deg}D$ depends only on the linear equivalence class of D as the first consequence of the Riemann-Roch Theorem. Since $r \in X(k)$ is determined uniquely by the linear equivalence class of $\langle r\rangle$, this formula tells us that $\varphi(D)$ depends only on the linear equivalence class of D. However, the formula also tells us that the linear equivalence class of D depends only on $\varphi(D)$ and $\mathrm{deg}(D)$. In particular, one has

$$\mathrm{Div}_0(X)/\mathrm{Div}_\ell(X) \cong X(k)\quad .$$

Wed., Apr. 26:

Suppose that X is a complete non-singular curve over an algebraically closed field k of genus 1. The range of degrees where a divisor D has $H^1(\mathcal{O}(D)) \cong (0)$ is $\deg(D) \geq 1$, while we have $\dim H^1(\mathcal{O}) = 1$. For each $a \in X(k)$ the invertible module $\mathcal{O}(2\langle a\rangle)$ has no base point, and, therefore, defines a morphism to $\mathbf{P}^1_k$. One has a two step filtration of the 3-dimensional linear subspace $L(3\langle a\rangle)$ of $k(X)$:

$$k \;=\; L(0) \;=\; L(\langle a\rangle) \subset L(2\langle a\rangle) \subset L(3\langle a\rangle)\quad .$$

Choosing $x \in L(2\langle a\rangle) - L(0)$ and $y \in L(3\langle a\rangle) - L(2\langle a\rangle)$ one obtains a filtration-compatible basis $\{1, x, y\}$ of $L(3\langle a\rangle)$, and if u is a "rational section" of $\mathcal{O}(\langle a\rangle)$ with $\mathrm{div}(u) = \langle a\rangle$, the morphism $f : X \to \mathbf{P}^2_k$ given by

$$f \;=\; (Z:X:Y),\quad Z \;=\; u^3,\ X \;=\; xu^3,\ Y \;=\; yu^3$$

provides a projective embedding of X by the theorem of the last hour. Extending the filtration inside $k(X)$ by the $L(m\langle a\rangle)$, one sees that $\{1, x, y, x^2, xy, x^3\}$ is a filtration-compatible basis of $L(6\langle a\rangle)$. Since $y^2 \in L(6\langle a\rangle) - L(5\langle a\rangle)$, one has a linear relation among monomials of degree 3

$$Y^2Z + a_1XYZ + a_3YZ^2 \;=\; a_0X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with $a_0 \neq 0$ that characterizes $f(X)$ as a non-singular hypersurface in $\mathbf{P}^2_k$. One says that $f(X)$ is in generalized Weierstrass form. One regards $Z = 0$ as the "line at infinity" in $\mathbf{P}^2_k$, while one calls "affine" a point $(X, Y) = (1 : X : Y)$. The intersection of $f(X)$ with the line at infinity reduces to the equation $a_0X^3 = 0$. Therefore, the point $(0 : 0 : 1)$ is the only point of $f(X)$ on the line at infinity, and as the point of intersection of the line at infinity with $f(X)$ it has multiplicity 3.

Mon., Apr. 24:

Continuing with the case of a complete normal curve over an algebraically closed field k. When D is a divisor with $\deg(D) \geq 2g$, then for each $a \in X(k)$ one has $\deg(D - \langle a\rangle) \geq 2g - 1$, and, therefore, $L(D - \langle a\rangle)$ is a hyperplane in $L(D)$. Otherwise, said $\mathcal{O}(D)$ has no base point. A coordinate-free interpretation of the morphism $f : X \to \mathbf{P}^N_k$, where $N = \deg(D) - g$, given by a basis of $H^0(X, \mathcal{O}(D))$ is that $f(a)$ is the hyperplane $H^0(X, \mathcal{O}(D - \langle a\rangle))$ regarded as a point in the projective space of hyperplanes through the origin in $H^0(X, \mathcal{O}(D))$. If, moreover, $\deg(D) \geq 2g + 1$, then for $a \neq b$ in $X(k)$ it follows that $H^0(X, \mathcal{O}(D - \langle a\rangle - \langle b\rangle))$ has codimension 2 in $H^0(X, \mathcal{O}(D))$ so that $f(a)$ and $f(b)$ must be different points, i.e., f is injective. Since X is complete, $f(X)$ must be a closed subvariety of dimension 1 in $\mathbf{P}^N_k$. The fact that $H^0(X, \mathcal{O}(D - 2\langle a\rangle))$ also has codimension 2 in $H^0(X, \mathcal{O}(D))$ guarantees that $d_a(f) : T_a(X) \to T_{f(a)}(\mathbf{P}^N_k)$ has rank 1 for each a, and, therefore, that $f(X)$ is itself a complete non-singular curve. Since morphisms of complete non-singular curves are dual to the contravariant function field extensions, f must be an isomorphism, i.e., $\mathcal{O}(D)$ is very ample when $\deg(D) \geq 2g + 1$. As first example, when $g = 0$ and $D = \langle a\rangle$, the morphism f given by $H^0(X, \mathcal{O}(\langle a\rangle))$ is an isomorphism of X with $\mathbf{P}^1_k$.

Fri., Apr. 21:

In the context of a complete normal variety X over an algebraically closed field k an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is called *very ample* if there is an integer $N \geq 0$ and a closed immersion $f : X \to \mathbf{P}^N_k$ such that $\mathcal{L} \cong f^*\mathcal{O}_{\mathbf{P}^N_k}(1)$. (Recall the earlier description of the functor of points over k of $\mathbf{P}^N_k$.) If $\mathcal{L}$ is very ample, then $\mathcal{L}^{\otimes m}$ is also very ample for each $m \geq 1$. One says that $\mathcal{L}$ is *ample* if there exists $m \geq 1$ such that $\mathcal{L}^{\otimes m}$ is very ample. Finally, if there is an integer $N \geq 0$ and a morphism $f : X \to \mathbf{P}^N_k$ such that $\mathcal{L} \cong f^*\mathcal{O}_{\mathbf{P}^N_k}(1)$, one says that $\mathcal{L}$ *has no base point.* For a particular value of N if $z_0, \ldots z_N$ are homogeneous coordinates in $\mathbf{P}^N_k$, hence, a basis of $H^0(\mathbf{P}^N_k, \mathcal{O}_{\mathbf{P}^N_k}(1))$, then the $s_j = f^*z_j$ are elements of $H^0(X, \mathcal{L})$ that do not vanish simultaneously at any point of $X(k)$. It follows that the members of any basis of $H^0(X, \mathcal{L})$ also have no common zero, but it does not follow that the $\{s_j\}$ form a basis.

When $\dim(X) = 1$, recall that for a divisor D of negative degree one has $\dim_k(H^0(X, \mathcal{O}(D))) = 0$. If K is a canonical divisor and D a divisor with

$$\deg(D) > \deg(K) \;=\; 2g - 2\;,$$

then $K - D$ is a divisor of negative degree, and, consequently, by Serre duality $\dim_k H^1(X, \mathcal{O}(D)) = 0$ for any divisor D with $\deg(D) \geq 2g - 1$. When the genus $g = 1$, this means that $\dim_k H^1(X, \mathcal{O}(D)) = 0$ for any divisor D of degree at least 1. The Riemann Roch formula then implies that $\dim H^0(X, \mathcal{O}(D)) = \deg(D)$. In particular if $D = \langle a\rangle$ for $a \in X(k)$, one sees that $L(\langle a\rangle) \supseteq L(0) \cong k$ while both have dimension 1. Hence, there can be no $f \in k(X)^*$ with only a single simple pole. The same type of reasoning shows that $k(X)^*$ contains an element whose only pole is a double pole at a given point $a \in X(k)$.

Wed., Apr. 19:

When A is a ring and B an A-algebra, the module $\Omega_{B/A}$ is the B-module receiving an A-derivation from B that is initially universal for derivations from B to B-modules. When $f : X \to Y$ is a morphism of schemes there is an $\mathcal{O}_X$-module $\Omega_{X/Y}$ that globalizes the module of differentials from commutative algebra. A morphism $f : X \to Y$ of irreducible varieties over an algebraically closed field k is called *smooth* if (i) f is dominant, i.e., $\overline{f(X)} = Y$, and (ii) $\Omega_{X/Y}$ is a locally-free $\mathcal{O}_X$-module of rank $\dim(X) - \dim(Y)$. A *non-singular* variety over k is a variety X that is smooth over k. (An irreducible variety of dimension 1 is non-singular if and only if it is normal.) When X is a non-singular variety, one defines Ω^p_X to be the p-th exterior power $\wedge^p \Omega_{X/k}$. For $n = \dim(X)$ the top exterior power $\omega_X = \Omega^n_X$ is a locally-free $\mathcal{O}_X$-module that is called the *canonical* $\mathcal{O}_X$-module.

A form of Serre duality, which could be the subject of an entire course, is this:

Theorem. *If X is a complete non-singular variety of dimension n and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then $H^p(X, \mathcal{F})$ and $Ext^{n-p}_{\mathcal{O}}(\mathcal{F}, \omega_X)$ are dual vector spaces over k.*

An important special case is that when $\mathcal{F}$ is a locally-free $\mathcal{O}$-module. Then

$$\mathrm{Ext}^{n-p}_{\mathcal{O}}(\mathcal{F}, \omega_X) \cong \mathrm{Ext}^{n-p}_{\mathcal{O}}(\mathcal{O}, \omega_X \otimes \mathcal{F}^\vee) \cong H^{n-p}(X, \omega_X \otimes \mathcal{F}^\vee)$$

where $\mathcal{F}^\vee$ denotes the $\mathcal{O}$ dual of $\mathcal{F}$. In the case of a complete normal curve a *canonical divisor* is any divisor K for which $\mathcal{O}(K) \cong \omega_X$. When $\mathcal{F} = \mathcal{O}(D)$ for an arbitary divisor D, the vector spaces $H^p(X, \mathcal{O}(D))$ and $H^{1-p}(X, \mathcal{O}(K - D))$ have the same dimension for $p = 0, 1$. In particular one has $g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_X)$, and application of the Riemann-Roch formula to a canonical divisor leads to the conclusion that any canonical divisor must have degree $2g - 2$.

Mon., Apr. 17:

Continuing with the case of a complete normal curve X over an algebraically closed field, some observations:

1. If $H^0(X, \mathcal{O}(D)) \neq (0)$, then $\deg(D) \geq 0$ since D is linearly equivalent to a non-negative divisor $\mathrm{div}(f) + D$ for some $f \in L(D)$.
2. The set
$$|D| \;=\; \{\, E \in \mathrm{Div}(X) \mid E \geq 0,\; E \equiv D \}$$
is called the *complete linear system* determined by D. It may be bijectively identified with the projective space of lines through the origin in the vector space $L(D) \cong H^0(X, \mathcal{O}(D))$. A *linear system* is a projective subspace of a complete linear system. One has
$$|D| \;=\; \left\{ \mathrm{div}(s) \mid s \in H^0(X, \mathcal{O}(D)) \right\} \quad .$$
3. Looking at the cohomology sequence associated with the short exact sequence
$$0 \to \mathcal{O}(D - \langle a \rangle) \to \mathcal{O}(D) \to i_* i^* \mathcal{O}(D) \to 0 \;,$$
one sees that in going from $D - \langle a \rangle$ to D either the dimension of H^0 goes up by 1 or the dimension of H^1 goes down by 1 but not both.
4. To go further with complete normal curves we want to talk about Serre duality.

Fri., Apr. 7:

When X is a complete normal curve over an algebraically closed field k, $a \in X$ a closed point, $\langle a \rangle$ the corresponding divisor, and $i : \{a\} \to X$ the corresponding closed immersion of a subvariety, one has the exact sequence of coherent $\mathcal{O}$-modules

$$0 \to \mathcal{I}_{\{a\}} \to \mathcal{O} \to i_* \mathcal{O}_{\{a\}} \to 0 \;,$$

and, remembering that $\mathcal{I}_{\{a\}} \cong \mathcal{O}(-\langle a \rangle)$, then tensoring this exact sequence with the invertible $\mathcal{O}$-module $\mathcal{O}(D)$, D an arbitary divisor on X, one obtains

$$0 \to \mathcal{O}(D - \langle a \rangle) \to \mathcal{O}(D) \to i_* i^* \mathcal{O}(D) \to 0 \quad .$$

The third term above is a skyscraper that is rank 1 on $\mathcal{O}_{\{a\}}(\{a\}) \cong k$. The relation among Euler characteristics given by the last short exact sequence reduces to

$$\chi(X, D) \;=\; \chi(X, D - \langle a\rangle) + 1$$

for every divisor D and every closed point $a \in X$, and, thus, the observation that $\chi(X,D) - \deg(D)$ is a constant depending only on X where

$$\deg(D) \;=\; \sum_z n_z \quad \text{when} \quad D \;=\; \sum_z n_z \langle z\rangle \quad .$$

This provides a substantial portion of the Riemann-Roch Theorem:

$$\chi(X, D) \;=\; \deg(D) + 1 - g$$

where g, the *genus* of X, is defined as $\dim_k H^1(X, \mathcal{O})$. As a corollary of this, together with the observation that $\chi(X, D)$ depends only on $\mathcal{O}(D)$, one sees that $\deg(D)$ depends only on $\mathcal{O}(D)$, and, therefore, $\deg(\mathrm{div}(f)) \;=\; 0$ for each $f \in k(X)^*$, a result that corresponds to the statement for compact Riemann surfaces that the number of zeroes of a meromorphic function equals the number of its poles.

For an initial understanding of the genus of a complete normal curve, consider the exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{O} \to \underline{k(X)} \to \underline{k(X)}/\mathcal{O} \to 0$$

from which ensues the sequence of vector spaces over k

$$0 \to k \to k(X) \to H^0(X, \underline{k(X)}/\mathcal{O}) \to H^1(X, \mathcal{O}) \to 0$$

where the last 0 is H^1 of the constant, hence flasque, sheaf $\underline{k(X)}$ and $H^0(X, \underline{k(X)}/\mathcal{O})$ is the vector space of "principal part specifications". Thus, $g \;=\; 0$ if and only if every principal part specification is realized by an element of $k(X)$. Thereby it is clear that the genus of $\mathbf{P}^1_k$ is 0.

Wed., Apr. 5:

For $D \in \mathrm{Div}(X)$, X a normal variety, one defines

$$L(D) \;=\; \{\, f \in k(X)^* \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\} \quad .$$

$L(D)$ is an $\mathcal{O}(X)$-module that is isomorphic to the module of global sections of $\mathcal{O}(D)$. While a (regular) section of a locally-free $\mathcal{O}$-module of rank 1 is not represented by a single element of $k(X)^*$, it does have local pieces that are unique up to multiplications from $\mathcal{O}^*$ and, consequently, has a globally well-defined divisor. If $s_f \neq 0$ is the section of $\mathcal{O}(D)$ corresponding biuniquely with $f \in L(D)$, one has $\mathrm{div}(s_f) \;=\; \mathrm{div}(f) + D$. One sees that $\dim_k H^0(X, \mathcal{O}(D)) > 0$ if and only if D is linearly equivalent to some non-negative divisor.

A non-negative divisor D determines an $\mathcal{O}$-ideal $\mathcal{I}_D$ that is locally the principal ideal generated by a local equation for D. It follows that $\mathcal{I}_D$ is a rank 1 locally-free $\mathcal{O}$-module, and one sees easily that it is isomorphic to $\mathcal{O}(-D)$.

When X is a complete variety over a field k and $\mathcal{M}$ a coherent $\mathcal{O}$-module the k-modules $H^q(X, \mathcal{M})$ are finite-dimensional over k for all q. This is a consequence of the more general fact that direct images and higher direct images of a coherent module under a proper morphism are coherent (see the text). One defines the *Euler characteristic* of a coherent $\mathcal{O}$-module by

$$\chi(X, \mathcal{M}) \;=\; \sum_{q=0}^{\dim(X)} (-1)^q \dim_k H^q(X, \mathcal{M}) \quad .$$

When

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is an exact sequence of coherent $\mathcal{O}$-modules on X, one has

$$\chi(X, \mathcal{M}) \;=\; \chi(X, \mathcal{M}') + \chi(X, \mathcal{M}'') \quad .$$

Mon., Apr. 3:

When X is a normal variety, the affine coordinate ring $\mathcal{O}(U)$ of an open affine subvariety U is the intersection of its localizations at the prime ideals corresponding to the irreducible closed sets in U of codimension 1. Hence $\mathcal{O}(X)^*$ is the kernel of the homomorphism div. Given a divisor $D \in \mathrm{Div}(X)$ and an open covering $\{U_i\}$ of X that principalizes D, say, $D|U_i = \mathrm{div}_{U_i}(f_i)$, it follows from the computation of the kernel of div on the open subvariety $U_{ij} = U_i \cap U_j$ that $f_i = u_{ij} f_j$ (all elements of $k(X)$) where $u_{ij} \in \mathcal{O}(U_{ij})^*$. The Cech 1-cocycle u_{ij} determines an element $\mathcal{O}(D)$ of the group $H^1_{\text{Cech}}(X, \mathcal{O}^*)$ of locally-free $\mathcal{O}$-modules of rank 1, the map $D \to \mathcal{O}(D)$ is a group homomorphism, and the sequence

$$1 \to \mathcal{O}(X)^* \to k(X)^* \to \mathrm{Div}(X) \to H^1_{\text{Cech}}(X, \mathcal{O}^*) \to 1$$

is exact. One says that two divisors D_1 and D_2 are *linearly equivalent* (and one may write $D_1 \equiv D_2$) if $D_2 - D_1 = \mathrm{div}(f)$ for some $f \in k(X)^*$ or, otherwise stated, if $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$.

Fri., Mar. 31:

For an irreducible variety X over an algebraically closed field k, a *divisor* is an element of the free abelian group $\mathrm{Div}(X)$ generated by the irreducible closed sets of codimension 1. When X is normal, the local ring at each irreducible closed set Z of codimension 1 is a principal valuation ring, and, therefore, each element $f \neq 0$ in the function field $k(X)$ gives rise to a divisor

$$\mathrm{div}(f) \;=\; \sum_Z \mathrm{ord}_Z(f) \;,$$

which is called a *principal divisor*. The map div : $k(X)^* \to \mathrm{Div}(X)$ is a homomorphism of abelian groups. Since an open set U in X is also a variety, the functor $U \to \mathrm{Div}(U)$ defines an abelian sheaf $\underline{\mathrm{Div}}$ on X that is easily seen to be flasque. When X is normal and Z an irreducible closed set of codimension 1, the divisor in an open neighborhood of Z of the unique prime in $\mathcal{O}_Z$ is the generating divisor corresponding to Z. Thus one sees that each divisor on X is locally principal.

Wed., Mar. 29:

If $f : X \to Y$ is an affine morphism of algebraic varieties over an algebraically closed field k, then for each quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ one has an isomorphism of $H^q(X, \mathcal{F})$ with $H^q(Y, f_*\mathcal{F})$. Finite morphisms and closed immersions present important special cases. To know the cohomology of every coherent $\mathcal{O}_P$-module on each projective space $P = \mathbf{P}^N_k$ is to know the cohomology of every coherent $\mathcal{O}_X$-module on every projective variety X.

Mon., Mar. 27:

On a Noetherian space the cohomological functor H^q for abelian sheaves vanishes when $q > \dim(X)$. The E_2 spectral sequence for composite functors is operative when application of the first functor to an injective object in its domain yields an object that is acyclic for the second functor. This applies to the direct image functor followed by the global sections functor on abelian sheaves since the direct image of an injective abelian sheaf is flasque.

Fri., Mar. 24:

On a Noetherian space (descending chain condition for closed sets) each of the sheaf cohomology functors H^q on the category of abelian sheaves commutes with direct limits.

Wed., Mar. 22:

More on cohomology: Every abelian sheaf on a topological space X may be regarded as a $\mathbf{Z}$-module (sheaf of modules over the constant sheaf $\mathbf{Z}$). As base cohomology one uses the derived functors of the global sections functor in the category of $\mathbf{Z}$-modules. An abelian sheaf is *flasque* if its restrictions between open sets are all surjective. Every flasque sheaf is acyclic for cohomology, and every injective $\mathcal{A}$-module, for any sheaf of rings $\mathcal{A}$ on X, is flasque. Consequently, sheaf cohomology in the category of $\mathcal{A}$-modules is consistent with that in the category of $\mathbf{Z}$-modules.

Mon., Mar. 20:

If $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a morphism of ringed spaces, for every $\mathcal{B}$-module $\mathcal{G}$ there is an $\mathcal{A}$-module pullback $f^*(\mathcal{G})$ which at stalk level satisfies

$$f^*(\mathcal{G})_x \;=\; \mathcal{G}_{f(x)} \otimes_{\mathcal{B}_{f(x)}} \mathcal{A}_x \quad .$$

For a morphism of affine schemes pullback of quasi-coherent modules on the target is the same thing as base extension. For $P = \mathbf{P}_k^N$, k an algebraically closed field, the exact sequence

$$\mathcal{O}_P^{N+1} \stackrel{(x_0,\ldots,x_N)}{\longrightarrow} \mathcal{O}_P(1) \to 0$$

given by

$$(f_0, \ldots, f_N) \mapsto f_0x_0 + \ldots + f_Nx_N$$

spawns, via pullback, the functor of points of $\mathbf{P}_k^N$ over k: a morphism $\varphi : X \to \mathbf{P}_k^N$ is “the same thing” as an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and an $N+1$-tuple of sections $s_0, \ldots s_N$ of $\mathcal{L}$ that do not “vanish” simultaneously, i.e., that provide the exact sequence

$$\mathcal{O}_X^{N+1} \stackrel{(s_0,\ldots,s_N)}{\longrightarrow} \mathcal{L} \to 0 \ ,$$

which is the φ-pullback of the referenced exact sequence on $\mathbf{P}_k^N$. For a k-valued point $x \in X(k)$ one has

$$\varphi(x) \;=\; (s_0(x) : s_1(x) : \ldots : s_N(x)) \quad .$$

Fri., Mar. 17:

The isomorphism classes of locally-free $\mathcal{A}$-modules of rank 1 form a group. The notion of an exact sequence of $\mathcal{A}$-modules. $\mathcal{A}$-modules form an abelian category in which every object admits an injective resolution. The global sections functor $\Gamma(\mathcal{M}) = \mathcal{M}(X)$ is left exact. The q-th cohomology functor $X \mapsto H^q(X, \mathcal{M})$ is defined as the q-th right derived functor of Γ. Sideline example: the short exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_{\text{hol}} \stackrel{e}{\to} \mathcal{O}^*_{\text{hol}} \to 0$$

of $\mathbf{Z}$-modules in complex analytic geometry, where $e(f) = e^{2\pi i f}$ is the complex exponential.

Wed., Mar. 15:

Homomorphisms of $\mathcal{A}$-modules when $\mathcal{A}$ is a sheaf of rings on a topological space. Locally-free $\mathcal{A}$-modules of rank r and transition matrices relative to a trivializing covering. An *invertible* $\mathcal{A}$-module is a locally-free $\mathcal{A}$-module of rank 1.

Mon., Mar. 13:

Class cancelled.

Fri., Mar. 10:

Properties and significance of the $\mathcal{O}_P$ modules $\mathcal{O}_P(d)$ on $P = \mathbf{P}_k^n$ for $d \in \mathbf{Z}$ where k is an algebraically closed field.

Wed., Mar. 8:

The concept of sheaf of modules on a ringed space. Quasi-coherent and coherent modules on a scheme. Examples.

Mon., Mar. 6:

If $f : X \to Y$ is a morphism of schemes with Y separated, then f is universally closed if every split base extension of f is closed. Proper morphisms. Valuative criteria for separated morphisms and proper morphisms.

Fri., Mar. 3:

Separated morphisms. If $f : X \to Y$ is an S-morphism and Y is separated over S, then the graph of f is closed in $X \times Y$ and f is separated if and only if X is separated over S. Henceforth, an algebraic variety will be assumed to be separated over its base field; consequently, all morphisms of varieties will be separated. In a scheme that is separated over an affine base, the intersection of any two open affines is affine.

Wed., Mar. 1:

If x is an element of X, the scheme underlying an irreducible algebraic variety, the Krull dimension of the local ring $\mathcal{O}_x$ is the codimension of $\overline{\{x\}}$ in X. When X is normal, the local ring at an irreducible subvariety of codimension 1 in X is a discrete valuation ring. The set of closed points of a complete and normal irreducible algebraic curve correspond biuniquely with the non-trivial discrete valuation rings in its function field that contain the ground field, and the entire structure of such a curve as a scheme may be recovered from its function field.

Mon., Feb. 27:

Finite morphisms — yet another class closed under composition and base extension. The normalization of an irreducible variety. Universally closed morphisms. Finite morphisms are universally closed.

Fri., Feb. 17:

Any base extension of a morphism of finite type is also a morphism of finite type. Case in point: the fibre of a morphism $f : X \to Y$ of finite type over an element $y \in Y$ is a scheme of finite type over the residue field $\kappa(y)$. Over its image a morphism may be viewed as providing a family of varieties, though not a well-behaved one without assumptions on the morphism. The notion of affine morphism: another class of morphisms that is closed under compostion and base extension.

Wed., Feb. 15:

The join of two Cartesian squares is another. Cartesian squares provide shelter for both the geometric notion of product and the algebraic notion of base extension. The notion of base extension of a morphism. Example: The action of $\mathrm{Gal}(\bar{k}/k)$ on $X_{\bar{k}}$ when X is a k-scheme (and $\bar{k}$ is the algebraic closure of the field k).

Mon., Feb. 13:

Detailed examination of the functor of points for $E = \mathrm{Spec}\,(\mathbf{Z}\,[x,y]\,/\,(F(x,y)))$ where $F(x,y)$ is the polynomial $F(x,y) = y^2 - (x-a)(x-b)(x-c)$, particularly in relation to base extensions of the coordinate ring. Existence and uniqueness of products in the category of schemes over a given scheme.

Fri., Feb. 10:

The notion of morphism of a scheme over a "base scheme" globalizes the notion of homomorphism for algebras over a base ring. If S is a scheme, the functor

$$(\mathrm{Schemes}/S)^{\mathrm{op}} \longrightarrow (\mathrm{Sets})$$

given by

$$T \longmapsto \mathrm{Hom}_S(T,X) \;=\; X(T)$$

is called the *functor of points* of X over S. X is determined as an S-scheme by its functor of points. If X is the scheme associated with a variety X_0 over an algebraically closed field k, then $X(k) = X(\mathrm{Spec}(k))$ is the set underlying X_0. If K is an extension field of k, a point $\xi \in X(K)$ determines an element $x \in X$ (no longer called a "point") that is called its *center* and a k-algebra homomorphism from the residue field at x to K. In the affine case $X(K)$ is precisely the set of naive points of X in K.

Wed., Feb. 8:

A morphism from a scheme to the affine scheme $\mathrm{Spec}(A)$ is dual to a ring homomorphism from A to the ring of global sections of the scheme's structure sheaf. The scheme associated with an affine variety over an algebraically closed field is characterized as a reduced scheme of finite type over (the spectrum of) the field.

Mon., Feb. 6:

The category of schemes. Locally closed subschemes. Morphisms; schemes over a base scheme.

Fri., Feb. 3:

The category of affine schemes as (1) a fully faithful subcategory of the category of local-ringed spaces and (2) as the opposite category of the category of commutative rings.

Wed., Feb. 1:

The notion of an affine scheme as a topological space equipped with a sheaf of rings; morphisms between affine schemes.

Mon., Jan. 30:

The sheaf of rings associated with the spectrum of a commutative ring; the initial ring is the ring of global sections.

Fri., Jan. 27:

The spectrum of a commutative ring and its Zariski topology.

Wed., Jan. 25:

Presheaves and sheaves; examples.

Mon., Jan. 23:

Overview.

2 Comments

Things Spotted on the Web

Wikipedia

There are a number of ways to enter.

- Algebraic Geometry[2]
- Schemes[3]
- Search *Wikipedia for "algebraic geometry"*[4]

Notes on Lectures by Hartshorne

These are notes by William Stein of 1996 lectures given by Robin Hartshorne at UC Berkeley: `http://modular.ucsd.edu/AG.html`.

UP | TOP | Department

[2]URI: http://en.wikipedia.org/wiki/Algebraic_Geometry
[3]URI: http://en.wikipedia.org/wiki/Scheme_%28mathematics%29
[4]URI: http://en.wikipedia.org/wiki/Special:Search?search=%22algebraic+geometry%22&fulltext=fulltext