# Advanced Linear Algebra (Math 424/524)
# Assignment No. 5

## Due December 11, 2002

1. Let $F$ be a field. For $P$ a polynomial in $F[t]$ let $V_P$ denote the quotient $F[t]/PF[t]$. Given $P$ and $Q$ in $F[t]$ one defines a linear map

$$\phi : V_{PQ} \longrightarrow V_P \times V_Q$$

by

$$\phi(h \bmod PQ) \;=\; (h \bmod P,\; h \bmod Q)\;.$$

What is the dimension of the kernel of $\phi$?

2. For each of the following *rational* matrices find the minimal and characteristic polynomials and find a direct sum of companion matrices that is similar to the given matrix:

$$\text{(a)}\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{(b)}\quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{(c)}\quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{(d)}\quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{(e)}\quad \begin{pmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & -2 & 1 & -1 \\ -1 & 3 & 1 & 0 & 0 \\ 2 & -1 & 3 & -1 & 2 \\ 1 & -1 & -3 & -2 & 0 \end{pmatrix}$$

3. Let $M$ be the matrix of polynomials in $F[t]$

$$M \;=\; \begin{pmatrix} t^3 + t^2 - 2t & t^3 - 2t + 1 \\ t^3 - 1 & t^3 - t^2 \end{pmatrix}\;.$$

   (a) Find a diagonal matrix of successively divisible polynomials that can be obtained from $M$ by (restricted) row and column operations.

   (b) What is the dimension of the quotient space

$$F[t]^2/MF[t]^2\ ?$$

4. Let $M$ be the matrix of polynomials from the previous problem.

   (a) Is there a $2 \times 2$ matrix $A$ for which $t \cdot 1 - A$ is (restricted) row and column equivalent to $M$?

   (b) Find an $N \times N$ matrix $A$ for some $N$ in the field $F$ for which the endomorphism "multiplication by $t$" of the quotient space in part (b) of the previous problem is isomorphic to the endomorphism of $F^N$ given by $A$.

5. For given $\lambda$ in the field $F$ let $J = J(\lambda)$ be the $r \times r$ matrix

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & & \vdots \\ \vdots & \vdots & & & & 0 \\ & & & & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

Find a basis for the vector space $F[t]/(t-\lambda)^r F[t]$ in which the matrix $J$ represents the endomorphism "multiplication by $t$".