The Classification of Isometries of $\mathbb{R}^3$: I

- Recall that every isometry of $\mathbb{R}^n$ is the composition of a translation and an isometry that fixes the origin, and that every isometry fixing the origin has the form $x \mapsto Ux$ where $U$ is an orthogonal matrix.

- There is a four-way division according to (a) whether an isometry is orientation-preserving or not and (b) according to whether it has a fixed point or not. But for $n = 3$ this does not give a complete description.

- The key to understanding the geometric structure of the isometry given by a $3 \times 3$ orthogonal matrix is to understand its eigenvectors and eigenvalues. Once that is done, one needs to analyze the transformation that results when one of those is followed by a translation.

- The characteristic polynomial of an $n \times n$ matrix $A$ is the determinant of the $n \times n$ matrix of polynomials $tI_n - A$ (with $t$ the variable). It is a polynomial of degree $n$ with leading coefficient 1 and constant term equal to $\det(-A) = (-1)^n \det(A)$.

- All of the eigenvalues of an orthogonal matrix must be of the form $a + ib$ where $a$ and $b$ are real with $a^2 + b^2 = 1$.

- Counting multiplicities, there are $n$ complex roots of any polynomial $f$ of degree $n \geq 1$. If the leading coefficient of the polynomial is 1, then the sum of its $n$ complex roots is the negative of the coefficient of degree $n - 1$, and the product of its $n$ complex roots is the constant term multiplied by $(-1)^n$.

- Since the characteristic polynomial of an orthogonal matrix is a polynomial with real coefficients, any of its roots that are not real must occur in complex-conjugate pairs. The product of any two complex-conjugate eigenvalues of an orthogonal matrix must be 1.

- Since the degree of the characteristic polynomial of a $3 \times 3$ matrix is odd, at least one of the eigenvalues of a $3 \times 3$ matrix must be real.

- **Proposition.** The three eigenvalues, counting multiplicities, of a $3 \times 3$ orthogonal matrix must be one of 1 or $-1$ and both of $\cos \theta \pm i \sin \theta$ for some real value of $\theta$, $0 \leq \theta < 2\pi$. If $\theta = 0$, then the latter two eigenvalues are both 1, and if $\theta = \pi$, then they are both $-1$.

Assignment for Monday, April 12

1. How may one describe the pencil of lines through the point $(3, -5, 2)$, (a point on the line at infinity represented by homogeneous coordinates relative to the affine basis $((1, 0), (0, 1), (0, 0))$) in terms of the ordinary Cartesian geometry of $\mathbb{R}^2$?

2. Let $f$ denote the linear isometry of $\mathbb{R}^3$ given by the formula $f(x) = Mx$ where $M$ is the $3 \times 3$ orthogonal matrix

$$M = \frac{1}{7} \begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix}.$$

Describe $f$ in geometric terms.

3. How much of the classification of the isometries of the plane would be obtained by pursuing a discussion for dimension 2 that is parallel to the discussion above for dimension 3?

4. Can a $3 \times 3$ orthogonal matrix other than the identity matrix be a scalar multiple of the affine matrix, relative to the affine basis $((1, 0), (0, 1), (0, 0))$, of an isometry of $\mathbb{R}^2$?