The Affine Matrix of an Affine Transformation

Recall from the study of linear algebra that if $f$ is a linear map from $\mathbb{R}^n$ to itself and $v = \{v_1, \ldots, v_n\}$ is a linear basis of $\mathbb{R}^n$, then the matrix of $f$ with respect to the basis $v$ is the $n \times n$ matrix $M$ whose $j$-th column, for $1 \leq j \leq n$, is the column of coordinates of $f(v_j)$ relative to $v$, i.e.,

$$f(v_j) = \sum_{i=1}^{n} M_{ij} v_i, \quad 1 \leq j \leq n.$$ 

**Definition.** If $p = \{p_0, \ldots, p_n\}$ is an affine basis of $\mathbb{R}^n$ and $f$ is an affine map from $\mathbb{R}^n$ to itself then the affine matrix of $f$ with respect to the affine basis $p$ is the $(n + 1) \times (n + 1)$ matrix $M$ whose $j$-th column, for $0 \leq j \leq n$, is the column of barycentric coordinates of $f(p_j)$ relative to $p$, i.e.,

$$f(p_j) = \sum_{i=0}^{n} M_{ij} p_i \quad \text{with} \quad \sum_{i=0}^{n} M_{ij} = 1, \quad 0 \leq j \leq n.$$ 

**Proposition.** If $p$ is a point of $\mathbb{R}^n$ having barycentric coordinates $(x_0, \ldots, x_n)$ relative to the affine basis $p$ and if $f$ is an affine map having matrix $M$ relative to $p$, then $f(p)$ is the point of $\mathbb{R}^n$ having barycentric coordinates $(y_0, \ldots, y_n)$ relative to $p$ where the vectors $x$ and $y$, when regarded as columns, are related by the formula $y = Mx$.

**Proof.** Because $f$ preserves barycentric combinations and $p = x_0 p_0 + \ldots x_n p_n$ with $x_0 + \ldots + x_n = 1$, it follows that

$$f(p) = \sum_j x_j f(p_j) = \sum_j x_j \left( \sum_i M_{ij} p_i \right) = \sum_{ij} M_{ij} x_j p_i = \sum_{i} \left( \sum_j M_{ij} x_j \right) p_i = \sum_i y_i p_i \quad \text{where} \quad y = Mx.$$ 

One needs to check that the last line is indeed a barycentric combination of the $p_i$, i.e., that $y_0 + \ldots + y_n = 1$. This follows from the fact that $y$ is the $x$-barycentric combination of the (weight 1) columns of $M$.

**Exercises due Monday, March 15**

1. Show that the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(x_1, x_2) \mapsto (x_1, x_2, 1 - x_1 - x_2)$ is an affine map.

2. Conclude from the first exercise that if $\tau$ is translation of $\mathbb{R}^2$ by the vector $a = (a_1, a_2)$, then $\varphi(\tau(x)) = \varphi(x) + \hat{a}$ where $\hat{a}$ is the weight 0 triple $(a_1, a_2, a_3)$ with $a_3 = -a_1 - a_2$.

3. (Continuing) Find the affine matrix of the translation $\tau$.

4. Find the affine matrix of the half turn of $\mathbb{R}^2$ about the point $c$, i.e., the affine transformation $x \mapsto 2c - x$.

5. Show that if $M$ is the affine matrix of the affine transformation $f(x) = Ux + v$ of $\mathbb{R}^2$, then $\det M = \det U$. 