Discussion

• **Theorem.** Every orientation-preserving isometry of \( \mathbb{R}^2 \) with a fixed point is a rotation.

  *Proof.*** Let \( f \) be a given orientation-preserving isometry of \( \mathbb{R}^2 \) with fixed point \( c \). Let \( \tau \) be “translation by \( c \), i.e., \( \tau(x) = x + c \). Then the isometry \( g = \tau^{-1} \circ f \circ \tau \) has the property \( g(0) = 0 \). Since \( g \) is an affine map that fixes the origin, \( g \) must be a linear transformation of \( \mathbb{R}^2 \) that is distance-preserving. Therefore, \( g(x) = Ux \) for some \( 2 \times 2 \) orthogonal matrix \( U \). By an exercise in the previous assignment \( U \) must be one of the matrices formed using \( \cos \theta \) and \( \sin \theta \) for some value of \( \theta \), and since \( g \) is orientation-preserving, \( \det U > 0 \) with the result that \( U \) must be the specific matrix

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Therefore, \( g \) is the rotation about the origin through the angle \( \theta \), and \( f \) is the rotation about the point \( c \) through the angle \( \theta \).

• **Theorem.** Every orientation-reversing isometry of \( \mathbb{R}^2 \) with a given fixed point is the reflection in some line containing the fixed point.

  *Proof.*** The argument is very similar to the preceding argument except that the \( 2 \times 2 \) orthogonal matrix \( U \) satisfies \( \det(U) < 0 \) since the isometry is orientation-reversing, and, therefore,

\[
U = \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix}
\]

which is the matrix of reflection in the line through the origin forming the angle \( \theta/2 \) with the positive first coordinate axis.

• **Proposition** Every rotation of \( \mathbb{R}^2 \) is the composition of the reflections in two lines passing through its center.

  *Proof.*** For example, let \( \sigma_1 \) be reflection in the horizontal line through the center and let \( \sigma_2 \) be reflection in the line through the center forming angle \( \theta/2 \) with the horizontal, where \( \theta \) is the angle of rotation about the center. Then \( \sigma_2 \circ \sigma_1 \) is the given rotation.

Exercises due Monday, February 23

1. Prove: If an isometry \( f \) of the plane is a rotation about the point \( p \), then for every point \( x \) in the plane \( p \) must lie on the perpendicular bisector of the line segment from \( x \) to \( f(x) \).

2. Show that every translation of \( \mathbb{R}^2 \) is the composition of the reflections in two parallel lines that are perpendicular to the direction of translation.

3. Show that the composition of a rotation with the reflection in a line through the center of the rotation is another such reflection.

4. Let \( A, B, C, \) and \( P \) be four points in a plane, no three of which are collinear. Let \( PA \) meet \( BC \) at \( D \), \( PB \) meet \( CA \) at \( E \), and \( PC \) meet \( AB \) at \( F \). Prove that \( D, E, \) and \( F \) are not collinear.