Discussion

- **Terminology Revision.** Any weight 1 linear combination of given points may be called a *barycentric combination* of those points, regardless of whether the coefficients are non-negative.

- **Definition.** A sequence of \( r + 1 \) points \( p_0, p_1, \ldots, p_r \) is called *barycentrically independent* if none of them is a barycentric combination of the others.

- **Examples.**
  1. Any two distinct points \( P, Q \) are barycentrically independent. If \( P \neq Q \), the set of barycentric combinations of \( P \) and \( Q \) is the line through \( P \) and \( Q \).
  2. Three points \( A, B, C \) are barycentrically independent if and only if none lies on the line determined by the other two. Thus, the vertices of a triangle are barycentrically independent.
  3. In \( \mathbb{R}^3 \) the four vertices of a tetrahedron are barycentrically independent.

- **Proposition.** A sequence of \( r + 1 \) points \( p_0, \ldots, p_r \) is barycentrically independent if and only for given \( a_0, \ldots, a_r \) and given \( b_0, \ldots, b_r \) with \( a_0 + \ldots + a_r = 1 \) and \( b_0 + \ldots + b_r = 1 \) the following statement is true:
  \[
a_0 p_0 + \ldots + a_r p_r = b_0 p_0 + \ldots + b_r p_r \quad \text{if and only if} \quad a_0 = b_0, \ldots, a_r = b_r.
  \]

- **Proof.** Obtain this from corresponding facts about linear independence.

- **Theorem.** If \( p_0, p_1, \ldots, p_n \) are barycentrically independent points of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), and \( q_0, q_1, \ldots, q_n \) are any points of \( \mathbb{R}^n \), then there is one and only one affine map \( f \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) for which \( f(p_0) = q_0, f(p_1) = q_1, \ldots, f(p_n) = q_n \).

  **Proof.** Use the fact that there is a unique linear map taking prescribed values at the members of a basis of \( \mathbb{R}^n \).

- **Theorem** If a map \( f \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) preserves barycentric combinations, then it must be an affine map.

  **Proof.** Use two facts: (1) an affine map that carries 0 to 0 must be linear, and (2) a linear map is always given by a matrix.

**Exercises due Friday, January 30**

1. Let \( A, B, C, \) and \( D \) be four points in the plane \( \mathbb{R}^2 \). Show that the polygonal path (sequence of line segments) from \( A \) to \( B \), from \( B \) to \( C \), then to \( D \), and back to \( A \) is a parallelogram if and only if \( A - B + C - D = 0 \).

2. Show that an affine transformation of the plane carries a parallelogram to a parallelogram.

3. Show that there is one and only one affine transformation of the plane carrying a given parallelogram to another given parallelogram in a given vertex-matching way.

4. Show that any affine transformation of the plane carries the point where the diagonals of a given parallelogram meet to the point where the diagonals of the image parallelogram meet.

5. Explain why an affine transformation of the 3-dimensional space \( \mathbb{R}^3 \) must always carry a tetrahedron to a tetrahedron.