Math 331 – Homework Assignment

April 19, 2002

Reading in the Text

§§ 10.4 – 10.5

Commutators

- **Definition.** If f and g are transformations of a set X, the **commutator** of f and g is the transformation $fgf^{-1}g^{-1}$. Sometimes the commutator of f and g is denoted by $[f, g]$.
- While $[f, g]^{-1} = [g, f]$ and, therefore, the inverse of any commutator is always a commutator, it is not necessarily true that the product of two commutators is a commutator.

Similarities.

- **Definition.** A transformation f of $\mathbf{R}^n$ is called a **similarity** if there is a positive scalar r such that for all points x, y in $\mathbf{R}^n$ one has the distance relation

$$d(f(x), f(y)) \quad = \quad r\, d(x, y) \quad .$$

  The number r is called the **scaling factor** of f.
- **Proposition.** When two similarities are composed, the scaling factor of the composite similarity is the product of the scaling factors of the two original similarities.
- **Theorem.** A transformation f of $\mathbf{R}^n$ is a similarity if and only if in a rectangular coordinate system it is given by the formula

$$f(x) \quad = \quad r\,Ux \; + \; b \; ,$$

  where r is a positive scalar, U is an $n \times n$ orthogonal matrix, and b is a point of $\mathbf{R}^n$.

Assignment for Monday, April 22

1. Prove the proposition stated above.
2. Prove the theorem stated above.
3. Show that the commutator of two affine transformations of $\mathbf{R}^n$ must always be orientation-preserving and volume-preserving but need not be an isometry.
4. Show that the commutator of any two similarities of $\mathbf{R}^n$ must be an isometry.
5. Show that for $r > 0$ and c a given point of $\mathbf{R}^n$ the formula

$$f(x) \;=\; (1-r)c \,+\, rx$$

   defines a similarity f with scaling factor r that has c as a fixed point and that commutes with every affine transformation of $\mathbf{R}^n$ that has c as a fixed point. This type of similarity is called a *dilatation*. (Note that inasmuch as this definition of f involves a barycentric combination of c and x, the transformation f thereby defined does not depend on the choice of a rectangular coordinate system. Use this observation to simplify your argument.)
6. Given three lines a, b, and c in the plane that intersect in pairs so as to delimit a triangle, construct a point p and a line m so that the product of the reflections in the three given lines satisfies the equation

$$\sigma_c \circ \sigma_b \circ \sigma_a \quad = \quad \sigma_m \circ h_p \; ,$$

   where σ_m is reflection in the line m and h_p is the half turn about the point p.