Selected Homework Exercise Solutions
Math 331, Transformation Geometry

February 6, 2002

P. 17, no. 6: Prove that if S is on segment $\overline{PR}$ and T is on segment $\overline{QR}$, the segments $\overline{PT}$ and $\overline{QS}$ intersect.

Response. The exercise is mainly meaningful when S and T are not endpoints of the segments on which they lie and when P, Q, R are not collinear. If that is the case, then there are numbers s, t with $0 < s, t < 1$ such that $S = (1-s)R + sP$ and $T = (1-t)R + tQ$. Moreover, a point on the line PT has the form $(1-x)P + xT$ for some x, while a point on the line QS has the form $(1-y)Q + yS$ for some y. The lines PT and QS meet if and only if there are numbers x, y for which the two previous expressions are equal. The question of whether such values of x, y exist (and, hence, the lines intersect) is addressed algebraically.

If those expressions are expanded using the formulas for S and T, then their equality becomes the relation

$$(1-x)P + xtQ + x(1-t)R \;=\; ysP + (1-y)Q + y(1-s)R \quad .$$

With the assumption that P, Q, R are not collinear, hence, barycentrically independent, the corresponding coefficients of P, Q, R in this relation must be equal. Hence,

$$1-x \;=\; sy \;, \quad tx \;=\; 1-y \;, \quad (1-t)x \;=\; (1-s)y \;.$$

Solving these equations simultaneously for x, y one finds

$$x \;=\; \frac{1-s}{1-st} \;, \quad y \;=\; \frac{1-t}{1-st} \;.$$

The fact that these solutions exist means that the lines PT and QS intersect. Moreover, from the fact that $0 < s, t < 1$ it is clear that $0 < x, y < 1$, and, therefore, that the point where the lines intersect is the intersection of the segments $\overline{PT}$ and $\overline{QS}$.

P. 31, no. 4: P is a point inside a given triangle ABC, and F is the point on the side AB where the line CP meets AB. D is the point of intersection with AC of the line through P parallel to BC, and E is the point of intersection with BC of the line through P parallel to AC. Prove that $|AF| \cdot |CD| \cdot |BC| = |BF| \cdot |CE| \cdot |AC|$.

Response. If the vertices A, B, C are arranged clockwise, then each of the sides of the triangle is divided into two segments by the points F, E, D. Each corresponding length a, b, c is then decomposed: $a = a' + a''$, $b = b' + b''$, and $c = c' + c''$, where $a' = |CE|$, $b' = |AD|$, and $c' = |BF|$. With this notation the task is to show that $ab''c'' = bc'a'$.

Let $P = uA + vB + wC$. Since the point F has unique barycentric coordinates with respect to A, B, C and is both a barycentric combination of the two points C, P and also a barycentric combination of the two points A, B, one sees that

$$F \;=\; \frac{u}{u+v}A + \frac{v}{u+v}B \quad .$$

Let $D = (1-s)C + sA$ and $E = (1-t)C + tB$. By the parallelogram law of addition

$$P \;=\; D + E - C \;,$$

which leads to a second barycentric expression for P relative to the three vertices:

$$P \;=\; sA + tB + (1-s-t)C \quad .$$

Hence, $s = u,\ t = v$, and, therefore,

$$a' \;=\; va, \quad a'' \;=\; (1-v)a, \quad b' \;=\; (1-u)b, \quad b'' \;=\; ub \;,$$

while

$$c' \;=\; \frac{u}{u+v}c, \quad c'' \;=\; \frac{v}{u+v}c \quad .$$

Thus,

$$ab''c'' \;=\; \frac{uv}{u+v}abc \;=\; bc'a' \quad .$$