Transformation Geometry — Math 331

April 19, 2004

The Classification of Isometries of R^3: V

Theorem. Every isometry of $\mathbf{R}^3$ belongs to exactly one of the types in the following chart. With each type is listed its orientation behavior (+ if orientation-preserving), the dimension of its fixed point set (with -1 for "empty"), and the minimum number of mirror reflections in the factorization of an isometry as the composition of mirror reflections. The fact that the listed pair of numbers is unique to each row guarantees that there is no overlapping of these classes.

Type	Orientation	Fixed Locus	Mirror Factors
Identity	$+$	3	0
Rotation ($\neq 1$)	$+$	1	2
Translation ($\neq 1$)	$+$	-1	2
Screw	$+$	-1	4
Mirror reflection	-	2	1
Reflective rotation	-	0	3
Mirror glide	-	-1	3

Corollary. Each of these classes of isometries is stabilized under conjugation by an arbitrary isometry.

Proof. Clearly, orientation behavior and the dimension of the fixed point set of an isometry is not changed when the isometry is conjugated by another isometry, regardless of the type of the conjugator. It is also clear that any conjugate of a mirror reflection by another isometry is a mirror reflection since only mirror reflections have fixed point loci of dimension 2. Therefore, the number of mirror factors is preserved under conjugation since if h is an isometry and γ is conjugation by h, i.e., $\gamma(f) = h \circ f \circ h^{-1}$, one has $\gamma(g \circ f) = \gamma(g) \circ \gamma(f)$. Since the numerical pair consisting of the dimension of the fixed point set and the number of mirror factors characterizes each class and is unchanged by conjugation, it follows that each class is stabilized by conjugation.

Assignment for Wednesday, April 21

1. Give a short argument that the inverse of an isometry of $\mathbf{R}^3$ belongs to the same class as the isometry.
2. What type of isometry of $\mathbf{R}^3$ always has order 2?
3. What type of isometry is obtained as $f \circ f$ when f is a mirror glide?
4. What types of isometry of $\mathbf{R}^3$ can have finite order, i.e., the inverse isometry is a positive power of the isometry?
5. What type of isometry of $\mathbf{R}^3$ is obtained by composing the mirror reflections in the four faces of a general tetrahedron?
6. Which classes of isometries of $\mathbf{R}^3$ are closed under composition?
7. Give an example of an affine transformation of the plane $\mathbf{R}^2$ that has order 3, i.e., $f^{-1} = f^2$, that is *not* an isometry of $\mathbf{R}^2$.