Transformation Geometry — Math 331

March 29, 2004

Is the Line at Infinity a Special Line?

The question of whether the line at infinity is special in some way arises from comparing the result about lines in the projective plane stabilized by a projective transformation in the case that the projective transformation arises from the affine matrix for an affine transformation with the earlier result about lines stabilized by an affine transformation.

The result that the line $a \cdot x = c$ is stabilized by the affine transformation $f(x) = Ux + v$ if and only if there is a non-zero scalar λ such that ${}^tUa = \lambda a$ with $v \cdot a = (1-\lambda)c$ may be reformulated as the more concise blocked matrix relation

$$\begin{pmatrix} {}^tU & 0 \\ {}^tv & 1 \end{pmatrix} \begin{pmatrix} a \\ -c \end{pmatrix} = \lambda \begin{pmatrix} a \\ -c \end{pmatrix},$$

which says that the vector of coefficients of the function $a_1x_1 + a_2x_2 - cx_3$ is an eigenvector of a 3×3 matrix. However, the matrix in this relation is not the affine matrix of the transformation f (nor its transpose), and the vector $(a_1, a_2, -c)$ is not the coefficient vector of the homogeneous equation for the line in $\mathbf{R}^2$ given by the equation $a_1x_1 + a_2x_2 - c = 0$.

Reconciliation of the two results about stabilized lines lies in recognizing that a projective transformation of $\mathbf{P}^2$ relates them.

Previously we have matched a point $p = (x_1, x_2)$ in $\mathbf{R}^2$ with its triple of barycentric coordinates relative to the affine basis $\{(1,0),(0,1),(0,0)\}$ thereby associating with p the point $\varphi(p) = (x_1, x_2, 1 - x_1 - x_2)$ in $\mathbf{P}^2$. On the other hand, one may also consider the point $\psi(p) = (x_1, x_2, 1)$ in $\mathbf{P}^2$, and one finds that φ and ψ, when written as columns, are related by the formula

$$\varphi(p) = C\psi(p) \quad \text{with} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Inasmuch as C is an invertible 3×3 matrix, it gives rise to a projective transformation γ of $\mathbf{P}^2$, but since its columns do not all sum to the same value, C is not the homogeneous matrix of an affine transformation of $\mathbf{R}^2$. Consistent with the observation that C is not the homogeneous matrix of an affine transformation, one sees easily that γ carries the line $x_3 = 0$ to the line $x_1 + x_2 + x_3 = 0$, which is the line at infinity. This suggests that from the viewpoint of projective geometry there is nothing special about the line at infinity.

More precisely, the line $x_1 + x_2 + x_3 = 0$ has been the line at infinity because it represents the set of points in $\mathbf{P}^2$ that do not correspond under φ to points of $\mathbf{R}^2$. Similarly, the line $x_3 = 0$ is the line at infinity relative to ψ.

Proposition. When C is the matrix above, the homogeneous matrix M of the affine transformation f of $\mathbf{R}^2$ defined by the formula $f(x) = Ux + v$ is given by the relation

$$M = C \begin{pmatrix} U & v \\ 0 & 1 \end{pmatrix} C^{-1}.$$

Proof. The proof is a straightforward calculation.

Corollary. The conjugating matrix C of the foregoing proposition links the two different results on calculating the lines stabilized by an affine transformation of $\mathbf{R}^2$.

Exercises due Wednesday, March 31

1. To what line in $\mathbf{P}^2$ does the projective transformation γ (above) carry the line $x_1 + x_2 + x_3 = 0$?
2. Explain how the points of the line $x_3 = 0$ in $\mathbf{P}^2$ correspond relative to ψ to classes of parallel lines in $\mathbf{R}^2$.
3. What "curve" in $\mathbf{P}^2$ given by a purely quadratic equation (all terms of degree 2) arises from the hyperbola $x_1x_2 = 1$ in $\mathbf{R}^2$ via (a) φ and (b) ψ?
4. Given a line in $\mathbf{P}^2$ is there a projective transformation that carries that line to the line at infinity?
5. An affine transformation depends on 6 parameters in the sense that it is given as $x \mapsto Ux + v$ where U involves 4 variables and v involves 2 variables. In this spirit ponder the following:
   (a) On how many parameters does a reflection depend?
   (b) On how many parameters does a rotation transformation depend?
   (c) On how many parameters does a projective transformation depend?