Transformation Geometry — Math 331

March 17, 2004

Parametric Form of a Line in the Projective Plane

The fact that a line in the projective plane has a homogeneous equation of the form $ax + by + cz = 0$ (where $(a, b, c) \neq (0, 0, 0)$) reflects fact that $\mathbf{P}^2$ has one point for each line through the origin in $\mathbf{R}^3$ and a line in $\mathbf{P}^2$ consists of the points in $\mathbf{P}^2$ corresponding to lines in $\mathbf{R}^3$ lying in a plane through the origin of $\mathbf{R}^3$. Just as a plane through the origin in $\mathbf{R}^3$, which is the same thing as a 2-dimensional linear subspace of $\mathbf{R}^3$, consists of the set of all linear combinations $c = sa + tb$ of two linearly independent vectors a, b in $\mathbf{R}^3$, i.e., a linear basis of the plane, the line in $\mathbf{P}^2$ through two different points a, b may be represented as the set of all linear combinations $sa + tb$ of homogeneous coordinate vectors in $\mathbf{R}^3$ for the two given points with $(s, t) \neq (0, 0)$. To view a line in $\mathbf{P}^2$ as the set of all linear combinations of two of its points a and b is to provide what is sometimes called a *parametric representation* of the line — with *parameters* s and t.

Note that every line in $\mathbf{P}^2$ other than the line at infinity (the line $x+y+z = 0$) meets the line at infinity in a single point. If neither of two points a and b lies on the line at infinity, then homogeneous coordinate vectors for those points may be chosen so that $a_1 + a_2 + a_3 = 1$ and $b_1 + b_2 + b_3 = 1$, and then each of the points on the line through a and b except its single point on the line at infinity may be represented as $sa + tb$ with $s + t = 1$, consistent with the fact that these points all correspond to the barycentric combinations of the points (a_1, a_2) and (b_1, b_2) in $\mathbf{R}^2$.

Exercises due Friday, March 19

1. Find a homogeneous equation for the line in $\mathbf{P}^2$ containing the points of $\mathbf{P}^2$ with homogeneous coordinates $(1, -2, 2)$ and $(2, -1, -1)$. What is the ordinary equation for this line as a line in $\mathbf{R}^2$?

2. Find a parametric representation for the line in $\mathbf{P}^2$ given by the homogeneous equation $6x + 11y + 9z = 0$.

3. Let $f(x) = Ux + v$ be the isometry of $\mathbf{R}^2$ given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} -3 \\ 4 \end{pmatrix} .$$

   (a) Find the affine matrix of f relative to the affine basis $\{(1, 0), (0, 1), (0, 0)\}$ of $\mathbf{R}^2$.
   (b) Find all fixed points of f.
   (c) Find all lines stabilized by f.
   (d) For each line stabilized by f provide the homogeneous equation for it as a line in $\mathbf{P}^2$.
   (e) How may one see that a line in $\mathbf{P}^2$ is a stabilized line for f by considering its homogeneous equation in relation to the affine matrix of f?
   (f) In $\mathbf{P}^2$ should the line at infinity be regarded as a line that is stabilized by f?

4. Let $f(x) = Ux + v$ be the order 2 affine transformation of $\mathbf{R}^2$ given by

$$U = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} -3 \\ 2 \end{pmatrix} .$$

   (a) Find the affine matrix of f relative to the affine basis $\{(1, 0), (0, 1), (0, 0)\}$ of $\mathbf{R}^2$.
   (b) Find all fixed points of f.
   (c) Find all lines stabilized by f.
   (d) For each line stabilized by f provide the homogeneous equation for it as a line in $\mathbf{P}^2$.
   (e) How may one see that a point in $\mathbf{P}^2$ is fixed by f from considering its homogeneous coordinate vector in relation to the affine matrix of f?
   (f) How may one see that a line in $\mathbf{P}^2$ is a stabilized line for f by considering its homogeneous equation in relation to the affine matrix of f?

5. Show by techniques of linear algebra that the non-orthogonal matrix U of the preceding problem is similar (under conjugation by an invertible matrix) to the orthogonal matrix U of the problem before it. Then explain why the affine transformation of the preceding problem cannot be conjugate to the affine transformation of the problem before it.