Transformation Geometry — Math 331

March 10, 2004

Homogeneous Coordinates and Homogeneous Equations for Lines

Recall that a barycentric combination of a sequence of points in $\mathbf{R}^n$ is a linear combination of the points in the sequence for which the sum of the coefficients is 1. Recall, moreover, that an affine basis of $\mathbf{R}^n$ is a sequence of $n+1$ points in $\mathbf{R}^n$ having the property that every point of $\mathbf{R}^n$ is uniquely a barycentric combination of the members of the sequence. For a given point and a given affine basis the coefficients of the basis members in the unique barycentric combination of them that represents the given point are called the *barycentric coordinates* of the point with respect to the affine basis.

We have seen that barycentric coordinates are useful because they give one a method of building arithmetic into affine geometry in a way that does not depend on what Cartesian coordinate system is being used for $\mathbf{R}^n$. This makes it reasonable to expect, for example, that the point where the angle bisectors of a triangle meet has a relatively simple — and memorable[1] — representation as a barycentric combination of the vertices of the triangle as well as to expect, as another example, that a transformation with a geometric description such as the order 2 symmetry in a point[2] a has a simple description ($x \mapsto 2a - x$).

The question arises how lines in the plane are described relative to barycentric coordinates.

How is the line $ax+by+c = 0$, $(a,b) \neq (0,0)$, expressed in barycentric coordinates relative to the affine basis $\{(1,0),(0,1),(0,0)\}$ of $\mathbf{R}^2$? Relative to this affine basis the point represented as (x,y) in Cartesian coordinates has barycentric coordinates (x,y,t) where $t = 1-x-y$. Therefore, the equation of the line becomes $ax+by+c(t+x+y) = 0$ or $(a+c)x+(b+c)y+ct = 0$.

Because the set of solutions of the last equation is unchanged if one multiplies its coefficient vector $(a+c, b+c, c)$ by a non-zero scalar, only the parallel class of the coefficient vector is relevant. For the parallel class of a vector in $\mathbf{R}^3$ to be meaningful that vector must not be $(0,0,0)$. But in order for such an equation to come from a line in $\mathbf{R}^2$ the equation in the barycentric coordinates x, y, t must have the form $px+qy+rt = 0$ where the three coefficients p, q, r are not all the same. If that is the case, one takes $c = r$, $a = p-r$, and $b = q-r$, and then the condition that p, q, r are not all the same is equivalent to the condition that $(a,b) \neq (0,0)$.

Finally, as observed previously[3], a point with a given vector (x,y,t) of barycentric coordinates may be recovered from any vector (x',y',t'), $x'+y'+t' \neq 0$, of homogeneous coordinates for the point relative to the affine basis since each of these vectors is a non-zero scalar multiple of the other and, in fact, $(x',y',t') = \lambda(x,y,t)$ when $\lambda = x'+y'+t'$. It is obvious that the equation $px+qy+rt = 0$ may be regarded as the equation of a line in homogeneous coordinates, as well as the equation of the same line in barycentric coordinates, provided only that the coefficients p, q, r are not all the same.

Proposition If P, Q, R are non-collinear points in $\mathbf{R}^2$, then a line in the plane is given in homogeneous coordinates (x,y,z) relative to the affine basis $\{P,Q,R\}$ of $\mathbf{R}^2$ by an equation of the form $px+qy+rz = 0$ where not all of the coefficients p, q, r are 0.

Proof. Since the question of what is a line is not affected by affine transformation, one may use the unique affine transformation of $\mathbf{R}^2$ carrying $(1,0)$ to P, $(0,1)$ to Q, and $(0,0)$ to R, thereby effectively reducing the assertion of the proposition to the discussion preceding its statement.

Exercises due Friday, March 12

1. Find homogeneous equations relative to an affine basis $\{A,B,C\}$ of the three medians of triangle $\Delta\, ABC$.

2. Relative to the affine basis $\{(1,0),(0,1),(0,0)\}$ of $\mathbf{R}^2$ find the Cartesian coordinates of the point where the line with homogeneous equation $4x+3y+6z = 0$ meets the line $6x+11y+9z = 0$. What is the intersection of the planes in $\mathbf{R}^3$ given by these two equations?

3. Repeat the previous exercise for the homogeneous equations $4x+3y+6z = 0$ and $3x+2y = 0$.

4. Repeat for $4x+3y+6z = 0$ and $2x+3y = 0$.

5. Find a 3×3 matrix M such that the glide reflection $(x,y) \mapsto (x+2, -y)$ may be represented barycentrically relative to the affine basis $\{(1,0),(0,1),(0,0)\}$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad .$$

[1] URI: tg040218.html

[2] In the case $n = 2$ the order 2 symmetry in a point is a half turn.

[3] URI: tg040211.html