Transformation Geometry — Math 331

March 8, 2004

Conjugation of Isometries

Definition. If f and g are invertible affine transformations of $\mathbf{R}^n$, the *conjugate of f by g* is the transformation $g \circ f \circ g^{-1}$.

When the context is clear, one sometimes abbreviates notation for conjugation by writing

$$g \cdot f \;=\; g \circ f \circ g^{-1} \quad .$$

One views this as g "acting on" f. That is, an affine transformation of $\mathbf{R}^n$ not only describes a motion of $\mathbf{R}^n$ but also a motion on the set of affine transformations of $\mathbf{R}^n$: the action of the set of affine transformations on itself by conjugation.

Theorem 1. If $f(x) = Ux + v$ is an affine transformation of $\mathbf{R}^n$ and $\tau(x) = x + a$ is translation by the vector a, then $f \cdot \tau$ is translation by the vector Ua.

Proof. $f^{-1}(x) = U^{-1}x - U^{-1}v$, and, therefore $y = \tau \circ f^{-1}(x) = U^{-1}x - U^{-1}v + a$. Then $(f \cdot \tau)(x) = Uy + v = x + Ua$.

Warning. Although when a translation is conjugated by an arbitrary affine transformation, the result is a translation, it is not true that the result of conjugating an isometry by an affine transformation is always an isometry.

For example, if f is the simple linear transformation $(x, y) \mapsto (2x, y)$ of $\mathbf{R}^2$ and ρ is rotation through the angle $\pi/2$ about the origin, i.e., $(x, y) \mapsto (-y, x)$, then $f \cdot \rho$ is the linear map $(x, y) \mapsto (-2y, x/2)$, which is not an isometry.

On the other hand, it is clear that the result of conjugating an isometry by an isometry is an isometry. Moreover:

Proposition 1. If f and g are affine transformations of $\mathbf{R}^n$ and the point c is a fixed point of f, then the point $g(c)$ is a fixed point of $g \cdot f = g \circ f \circ g^{-1}$.

Proof. The proof is left as an exercise.

Proposition 2. If f and g are affine transformations of $\mathbf{R}^n$, then f is orientation-preserving if and only if $g \cdot f = g \circ f \circ g^{-1}$ is orientation-preserving.

Proof. If f has matrix part U, i.e., $f(x) = Ux + v$ for some v, then the matrix part of $g \cdot f$ is the matrix BUB^{-1} where B is the matrix part of g. Therefore, the matrix parts of f and of $g \cdot f$ have the same determinant, and so the question of orientation-preservation, which depends on the sign of the determinant, is the same for f and $g \cdot f$.

Theorem. Each of the four classes of isometries of the plane is carried to itself under conjugation by an isometry.

Proof. For isometries of the plane the four classes are determined by two property switches: (1) whether or not there is a fixed point and (2) whether or not the isometry is orientation-preserving. The two propositions above show that these questions are not affected by conjugation.

Exercises due Wednesday, March 10

1. Find a rotation of $\mathbf{R}^2$ that fails to be conjugated into an isometry by the affine transformation $(x, y) \mapsto (x + y, y)$.
2. Find a reflection of $\mathbf{R}^2$ that fails to be conjugated into an isometry by the affine transformation $(x, y) \mapsto (x + y, y)$.
3. Prove Proposition 1 above.
4. Show that if a rotation of the plane is conjugated by an isometry of the plane, then the resulting rotation either involves the same angle or its negative.
5. Inasmuch as each translation of $\mathbf{R}^n$ is given by a unique vector, the set of all translations of $\mathbf{R}^n$ may be regarded as an abstract vector space in which vector addition is composition of translations and for each scalar c and each translation τ the translation $c\tau$ is defined as the map $x \mapsto (1 - c)x + c\tau(x)$. Explain why, for a given affine transformation f, the map $\tau \mapsto f \cdot \tau$ is a linear transformation of the vector space of translations.