Transformation Geometry — Math 331

March 5, 2004

How an affine transformation moves a line

If l is a line in $\mathbf{R}^2$ and f an affine transformation of $\mathbf{R}^2$, then $f(l)$ denotes the line of all points $f(x)$ as x varies in l. A line in $\mathbf{R}^2$ is the set of all solutions x of a first degree equation $a_1x_1 + a_2x_2 = c$, or $a \cdot x = c$, where $a \neq (0,0)$. The line given by that equation coincides with the line $b_1x_1 + b_2x_2 = d$ if and only if the coefficient vectors are parallel, i.e., the vector (b_1, b_2, d) is a non-zero scalar multiple of (a_1, a_2, c)

Definition. A line l is stabilized by f if $f(l) = l$.

Proposition. If $f(x) = Ux + v$ is an affine transformation of $\mathbf{R}^n$ and H is the set of solutions of the equation $a \cdot x = c$ where a is a non-zero vector, then the set $f(H)$ of all points $f(x)$ as x varies in H is the set of solutions of the equation $b \cdot x = d$ where $b = {}^tU^{-1}a$ and $d = b \cdot v + c$.

Notes. (1) While H is a line if $n = 2$, H is a plane if $n = 3$. (2) A dot product $x \cdot y$ may be written as the matrix product ${}^txy$ of the row ${}^tx$ and the column y.

Proof. One needs to obtain the condition that a point x lies in $f(H)$ as an equation. In fact, x lies in $f(H)$ if and only if $f^{-1}(x) = U^{-1}x - U^{-1}v$ lies in H. Thus, x is in $f(H)$ if and only if ${}^taf^{-1}(x) = c$ or ${}^taU^{-1}x = {}^taU^{-1}v + c$.

Corollary. If $f(x) = Ux + v$ is an affine transformation of $\mathbf{R}^2$ and l is the line $a \cdot x = c$ where a is a non-zero vector, then f stabilizes l if and only if there is a non-zero scalar λ such that ${}^tUa = \lambda a$ and $a \cdot v = (1 - \lambda)c$.

Exercises due Monday, March 8

1. What happens to the line $x + y = 1$ under the affine transformation f given by the formula
$$f(x) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} ?$$
2. If f is an isometry, the condition ${}^tUa = \lambda a$ in the corollary is equivalent to the condition $Ua = \lambda^{-1}a$. Why?
3. Can a non-identity rotation of the plane stabilize a line?
4. Can an orientation-reversing order 2 affine transformation of the plane stabilize a line it does not fix?
5. Apply the corollary above to determine all lines stabilized by a glide reflection.
6. Show that a line is stabilized by an affine transformation f if and only if it is stabilized by f^{-1}.
7. In exercise 3 of the assignment due March 1 it is shown that the isometry obtained by composing reflections in the three sides of a triangle is a glide reflection. Combine the result of that exercise with those of the preceding two exercises in order to be able to say exactly what is the axis of this glide reflection.