Transformation Geometry — Math 331

February 2, 2004

Discussion

- **Formulas from linear algebra.**

$$x \cdot y \;=\; x_1 y_1 + \ldots + x_n y_n\,,\quad ||x|| \;=\; \sqrt{x \cdot x}\,,\quad d(x,y) \;=\; ||x-y||\,,\quad \cos\angle(x,y) \;=\; \frac{x \cdot y}{||x||\,||y||}$$

- **Definition.** A square matrix U is *orthogonal* if it is inverted by its transpose ${}^tU$.

- **Theorem.** Let U be an $n \times n$ matrix, and let f be the linear map from $\mathbf{R}^n$ to itself given by $f(x) = Ux$. Then f is distance preserving, i.e., $d(f(x), f(y)) = d(x,y)$ for all x, y if and only if f is dot product preserving, i.e., $f(x) \cdot f(y) = x \cdot y$ for all x, y if and only if U is orthogonal.

  Proof. Linear algebra.

- **Corollary.** An affine transformation $f(x) = Ux + v$ of $\mathbf{R}^n$ is distance-preserving if and only if its associated matrix U is an orthogonal matrix.

  Proof. f is the composition of a linear map and a translation, and all translations are distance preserving; therefore, f is distance preserving if and only if the linear map $x \mapsto Ux$ is distance preserving.

- **Terminology.** Synonyms for "distance-preserving affine transformation": *congruence*, *isometry*, *rigid motion*.

Exercises due Wednesday, February 4

1. How many different affine transformations of the plane permute the vertices of a given triangle?

2. How many of the 24 permutations of the vertices of a non-rectangular parallelogram may be realized with an affine transformation?

3. Let $f(x) = Ax$ be the linear transformation of the plane where A is the matrix

$$A \;=\; \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \quad .$$

   (a) What points x of the plane are "fixed" by f, i.e., satisfy $f(x) = x$?

   (b) What lines in the plane are carried by f to other lines?

   (c) What lines L in the plane are "stabilized" by f, i.e., satisfy the condition that $f(x)$ is on L if x is on L?

4. Let $P_0, P_1 \ldots P_n$ be $n+1$ given points in $\mathbf{R}^m$. Show that these points are barycentrically independent if and only if no non-trivial weight 0 linear combination of them vanishes, i.e., one has

$$\left[\sum_j c_j P_j \;=\; 0 \text{ and } \sum_j c_j \;=\; 0\right] \quad \text{if and only if} \quad [c_0 \;=\; c_1 \;=\; \ldots \;=\; c_n \;=\; 0] \quad .$$

5. Although the three perpendicular bisectors of the sides of a triangle meet in a point, the barycentric coordinates of that point need not all be the same, yet the barycentric coordinates of the three points where the bisectors meet the sides of the triangle, relative to the endpoints of the sides, are all 1/2. Why doesn't this contradict the principle of preservation of proportionality in barycentric coordinates?