Transformation Geometry — Math 331

January 23, 2004

Discussion

- Definition. A map from N-dimensional Cartesian space $\mathbf{R}^N$ to itself is called a **transformation** if it is bijective, i.e., both "one-to-one" and "onto". (This definition does not require "continuity" although definitions in other contexts might.)
- Definition. A map from N-dimensional Cartesian space $\mathbf{R}^N$ to M-dimensional Cartesian space $\mathbf{R}^M$ is an **affine map** if it has the form

  $$f(x) \;=\; Ux + v \;,$$

  where U is an $M \times N$ matrix and v is a point of $\mathbf{R}^M$.
- Proposition. An affine map preserves barycentric combinations in the following sense:

  $$\text{If } x \;=\; \sum_j u_j x_j \quad \text{with} \quad \sum_j u_j \;=\; 1\,, \quad \text{then} \quad f(x) \;=\; \sum_j u_j f(x_j) \quad .$$

- Definition. An **affine transformation** of N-dimensional Cartesian space $\mathbf{R}^N$ is a transformation that is also an affine map.
- Theorem. An affine map from $\mathbf{R}^N$ to itself is an affine transformation if and only if the associated $N \times N$ matrix is an invertible matrix. In this case the inverse map is an affine transformation whose associated matrix is the inverse of the matrix associated with the original affine transformation.

Exercises due Monday, January 26

Let A, B, C, and D be the points in the Cartesian plane that are given by

$$A \;=\; (0,-1)\;, \quad B \;=\; (3,4)\;, \quad C \;=\; (-1,1)\;, \quad \text{and} \quad D \;=\; (1,2)\;,$$

and let T be the triangle with vertices A, B, and C.

1. How does one decide by analytic methods based on representation of points in the plane by weight 1 linear combinations of three given non-collinear points if a point lies inside the triangle having those given points as vertices?
2. Find the barycentric coordinates of the point D with respect to the vertices of the triangle T. Does D lie inside T?
3. Find an affine transformation f of $\mathbf{R}^2$, i.e., find a matrix U and a vector v such that $f(x) = Ux + v$ for all x in $\mathbf{R}^2$, such that $f(0,0) = A$, $f(1,0) = B$, and $f(0,1) = C$.
4. Is more than one solution of the preceding exercise possible? For your solution f find (u,v) in $\mathbf{R}^2$ such that $f(u,v) = D$.
5. Let f and g be the affine maps from the Cartesian plane $\mathbf{R}^2$ to itself defined by

   $$f(x) \;=\; Rx + r \quad \text{and} \quad g(x) \;=\; Sx + s \quad .$$

   Compute $g \circ f$ and $f \circ g$, where '$\circ$' denotes *composition* of maps.
6. Could the plane be replaced by $\mathbf{R}^3$ in the preceding exercise? What about $\mathbf{R}^N$?
7. Prove that an affine map from $\mathbf{R}^N$ to $\mathbf{R}^M$ is a *linear map* (in the sense of "linear algebra") if and only if it carries the origin of $\mathbf{R}^N$ to the origin of $\mathbf{R}^M$.
8. Find the point where the three angle bissectors of T meet.