Written Assignment No. 5

due Wednesday, December 7, 2005

General Directions: Written assignments should be submitted typeset. What you submit must represent your own work.

Assigned Exercises

1. Decompose the polynomial $t^{12} - 1$ into irreducible factors in the ring $(\mathbb{Z}/5\mathbb{Z})[t]$.
2. Decompose the polynomial $t^8 - 1$ into irreducible factors in the ring $(\mathbb{Z}/2\mathbb{Z})[t]$.
3. Let $G$ denote the ring $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$ of Gaussian integers.
   (a) Find the set $G^*$ of all units in $G$.
   (b) Find a greatest common divisor in $G$ for 2 and $5 - \sqrt{-1}$.
4. Let $R$ denote the ring $\mathbb{Z} + \mathbb{Z}\sqrt{-5}$. Explain why 14 and $6 + 2\sqrt{-5}$ have no greatest common divisor in $R$. Hint: Look at the norms of these elements.
5. Let $m \geq 0$ be an integer, and let $R$ denote the ring $\mathbb{Z} + \mathbb{Z}\sqrt{-5}$. Let $T_m$ denote the additive subgroup of $R$ given by

   $$T_m = \mathbb{Z} \cdot 7 + \mathbb{Z} \cdot (m - \sqrt{-5})$$

   (a) Find the smallest value of $m \geq 0$ for which $T_m$ is an ideal in $R$.
   (b) Find a familiar ring that is isomorphic to the quotient ring $R/T_m$ for the value of $m$ obtained in the previous part.