Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/

February 21, 2008

1 Basic Concepts in Linear Algebra

- abstract *vector space*
- *linear subspace* of a vector space
- *linear map*
- *translation* in a vector space
- *image* or *range* of a map (linear or not)
- *fiber* or *pre-image* of a point under a map (linear or not)
- *kernel* or *nullspace* of a linear map
- *affine subspace* of a vector space

2 What is a Vector Space?

A *vector space* is a set equipped with an operation called vector addition and with a given meaning for “multiplication by a scalar” (multiplication of an element by a number) subject to the following axioms:

1. $(x + y) + z \,=\, x + (y + z)$
2. $x + y \,=\, y + x$
3. $x + \vec{0} \,=\, x$
4. $x + (-1)x \,=\, \vec{0}$
5. $(ab) \cdot x \,=\, a \cdot b \cdot x$
6. $a \cdot (x + y) \,=\, a \cdot x + a \cdot y$
7. $(a + b) \cdot x \,=\, a \cdot x + b \cdot x$

8. $1 \cdot x = x$
9. $0 \cdot x = \vec{0}$

3 Examples of Vector Spaces

- n-dimensional Cartesian space $\mathbf{R}^n$
- The plane in $\mathbf{R}^3$ consisting of all points (x, y, z) satisfying the equation $x - 3y + 2z = 0$
- The set of all quadratic polynomials in the variable t, i.e.,

$$\left\{at^2 + bt + c\right\}$$

- The set of all functions of the variable t having the form

$$c_1 \cos t + c_2 \sin t$$

- The set of all differentiable functions on $\mathbf{R}$
- The set of all continuous functions on the interval $0 \le t \le 1$
- The set of all solutions of the differential equation

$$y'' - 3y' + 2y = 0$$

4 What is a Linear Subspace?

A *linear subspace* of a vector space is a subset of the vector space having the following properties:

1. The sum of any two members of the subset is in the subset.
2. The element $\vec{0}$ of the vector space is in the subset.
3. The multiple of any element of the subset by any scalar is in the subset.

5 Examples of Subspaces

- Any line or plane through $\vec{0} = (0,0,0)$ in $\mathbf{R}^3$ is a linear subspace of $\mathbf{R}^3$.
- The set of all functions of the variable t having the form

  $$c_1 \cos t + c_2 \sin t$$

  is a linear subspace of the set of all differentiable functions on $\mathbf{R}$.

6 What is a Linear Map?

A map $V \stackrel{\phi}{\to} W$ from a vector space V to a vector space W is a *linear map* (or *linear transformation*) if it preserves linear combinations. This means that it satisfies the following axioms:

1. $\phi(x+y) = \phi(x) + \phi(y)$ for all x, y in V
2. $\phi(ax) = a\phi(x)$ for all scalars a and all x in V

Equivalent: (preservation of linear combinations):

$$\phi(ax+by) = a\phi(x) + b\phi(y)$$

for all scalars a and b and all elements x and y of V.

7 Examples of Linear Maps

- The map $\mathbf{R}^n \stackrel{f_M}{\to} \mathbf{R}^m$ given by an $m \times n$ matrix, i.e.,

  $$f_M(x) = Mx ,$$

  is a linear map:

  1. $M(x+y) = Mx + My$
  2. $M(ax) = aMx$ (for every scalar a)

- Differentiation is linear:

  1. $D(f+g) = Df + Dg$
  2. $D(cf) = cDf$ (for every constant c)

8 What is a Translation?

A map $V \xrightarrow{T} V$ from a vector space V to itself is called a *translation* if it has the form

$$T(x) \;=\; x + v$$

for some fixed v in V. The particular translation given by an element v of V is called *translation by* v.

9 Translations vs. Linear Maps

For any vector space V the translation T_v by v is linear if and only if $v \;=\; \vec{0}$. In this case it is the identity map from V to V

$$T_{\vec{0}}(x) \;=\; x \quad .$$

That is, for a given vector space V the only map from V to V that is both a linear map and a translation is the identity map.

10 Image of a Map

If $X \xrightarrow{f} Y$ is a map from a set X to a set Y the *image* or *range* of f is the subset of the target Y of f consisting of all y in Y for which there is at least one x in X such that $y \;=\; f(x)$.
Notation: $f(X)$ may be used to denote the image of f.
The image of a linear map between vector spaces is its image as a map between sets.
Theorem. If $V \xrightarrow{\phi} W$ is a linear map, then $\phi(V)$ is a linear subspace of W.

11 Fiber of a Map Over a Point

If $X \xrightarrow{f} Y$ is a map from a set X to a set Y and y is an element of Y, the *fiber of* f over y is the set $f^{-1}(y)$ consisting of all x in X such that $f(x) \;=\; y$.
Tautology: The image of f is the set of all y in Y for which the fiber $f^{-1}(y)$ is not empty.

12 Kernel of a Linear Map

Definition. The *kernel* or *nullspace* of a linear map $V \xrightarrow{\phi} W$ is its fiber $\phi^{-1}(\vec{0}_W)$ over the origin of its target.

Re-stated: Kerϕ is the set of all v in V such that $\phi(v) = 0$.

Notation: Kerϕ may denote the kernel of ϕ.

Notes:

- The origin $\vec{0} = \vec{0}_V$ of V is always an element of the kernel of ϕ
- Sometimes, but certainly not always, the kernel of ϕ contains only $\vec{0}$.

Theorem. If $V \xrightarrow{\phi} W$ is a linear map, then Kerϕ is a linear subspace of V.

13 Significance of the Kernel

Every non-empty fiber of a linear map is a translate of its kernel.
Theorem. If $V \xrightarrow{\phi} W$ is a linear map and if v is a member of $\phi^{-1}(w)$, i.e., $\phi(v) = w$, then

$$\phi^{-1}(w) = T_v(\mathrm{Ker}\phi) = v + \mathrm{Ker}\phi \quad .$$

In other words, the most general x in V satisfying $\phi(x) = w$ has the form $x = v + z$ where z is the most general member of the kernel of ϕ.

14 What is an Affine Subspace?

An *affine subspace* of a vector space V is a subset of V that is a translate of a linear subspace.
Examples.

- Every line or plane in $\mathbf{R}^3$
- The set of solutions of any system of linear equations
- Every non-empty fiber of a linear map

Every linear subspace of a vector space is automatically an affine subspace.

15 Challenging Exercise

A non-empty subset S of a vector space is an affine subspace if and only if for all x and y in S and all scalars a and b with $a+b = 1$ the element $ax+by$ is a member of S.

16 To find a definition in a book

Try the index

17 Finding a definition in *Wikipedia*

Example: the term *fiber*
Google for: `site:wikipedia.org + fiber`
Find `http://en.wikipedia.org/wiki/Fiber`
Use disambiguation link at top of that page with option for "fiber (mathematics)" to find `http://en.wikipedia.org/wiki/Fiber_%28mathematics%29`