Math 220 Class Slides

February 7, 2008

1 Assignment due Jan 31, Nos. 1 & 3

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 5 & -4 & 3 \\ 3 & -3 & 2 \end{pmatrix}$$

Recall that

$$Y = MX \quad \text{if and only if} \quad X = QY$$

where

$$Q = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -2 \\ 3 & 0 & -1 \end{pmatrix}$$

Q is the inverse matrix of M

2 How was the Inverse Matrix Q Found?

1. Form the augmented matrix (MY) with Y general
2. Perform row operations on (MY) to bring its coefficient matrix portion into reduced row echelon form.
3. **IF** the coefficient portion of the new matrix is the identity matrix, then M is invertible.
4. **IF** the coefficient portion of the new matrix is the identity matrix, then extract Q as the matrix of coefficients in the last column of the RREF.

3 Another Way to Find the Inverse

1. If M has size $n \times n$, form the hyper-augmented matrix (MI_n) of size $n \times 2n$, where I_n is the $n \times n$ identity matrix.
2. Perform row operations on (MI_n) to bring its first n columns into reduced row echelon form.
3. **IF** the first n columns of the new matrix form I_n, then
   (a) M is invertible.
   (b) the last n columns of the new matrix form the inverse matrix Q.

4 Example

To find the inverse of the matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

Form the 2×4 matrix

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Then perform row operations

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$(R_1 \leftrightarrow R_2) \begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

$$(R_1 \rightarrow -R_1) \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

$$(R_2 \rightarrow R_2 + R_1) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

5 The Assignment: No. 1

$$\mathbf{R}^4 \xrightarrow{f} \mathbf{R}^3 \qquad f(X) = f_A(X) = AX$$

where

$$A = \begin{pmatrix} 2 & 3 & 1 & -4 \\ 3 & -2 & -1 & 5 \\ 5 & 1 & 0 & 1 \end{pmatrix}$$

- To handle all three tasks perform row operations on the augmented matrix

$$\begin{pmatrix} 2 & 3 & 1 & -4 & y_1 \\ 3 & -2 & -1 & 5 & y_2 \\ 5 & 1 & 0 & 1 & y_3 \end{pmatrix}$$

- The reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -1/13 & 7/13 & (2y_1+3y_2)/13 \\ 0 & 1 & 5/13 & -22/13 & (3y_1-2y_2)/13 \\ 0 & 0 & 0 & 0 & y_3-y_1-y_2 \end{pmatrix}$$

- There are no solutions unless $y_3 = y_1 + y_2$.
- When $y_3 = y_1 + y_2$, the transformed system of linear equations is:

$$\begin{aligned} x_1 - \frac{1}{13}x_3 + \frac{7}{13}x_4 &= \frac{2y_1+3y_2}{13} \\ x_2 + \frac{5}{13}x_3 - \frac{22}{13}x_4 &= \frac{3y_1-2y_2}{13} \end{aligned}$$

- x_1 and x_2 may be expressed as functions of x_3 and x_4.
- The variables corresponding to pivot columns may be expressed as functions of the other variables.
- Every solution for the case when $Y = 0$ has the form

$$Z = s\begin{pmatrix} 1/13 \\ -5/13 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -7/13 \\ 22/13 \\ 0 \\ 1 \end{pmatrix}$$

  as the parameters s and t range over all real values — a “plane” in the 4-dimensional space $\mathbf{R}^4$.
- The set of Y for which $f(X) = Y$ has at least one solution — the image of f — is the plane in $\mathbf{R}^3$ given by the equation $y_1 + y_2 - y_3 = 0$.
- Every solution W of

$$AX = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

  has the form

$$W = Z + \begin{pmatrix} 5/13 \\ 14/13 \\ 0 \\ 0 \end{pmatrix}$$

  where Z is given by the formula above.