Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/
Course Assignments Slides

March 11, 2008

1 Reminder

Midterm Test

next Tuesday

March 18

2 March 6: Exercise No. 1

- **Task:** If possible, invert the 4×4 matrix

$$M = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -2 & -1 & 3 & 2 \\ -2 & 2 & 6 & -1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

- Form the 4×8 matrix

$$\begin{pmatrix} M & 1_4 \end{pmatrix}$$

that augments M with the 4×4 identity matrix 1_4, and use row operations to maneuver the first 4 columns of that into reduced row echelon form.

- In this case the RREF of the first 4 columns is 1_4 so the last 4 columns of the reduced matrix form the inverse of M, which is:

$$M^{-1} = \begin{pmatrix} 2 & -4 & -4 & -17 \\ -1 & 7/3 & 8/3 & 11 \\ 1 & -2 & -2 & -9 \\ 0 & 2/3 & 1/3 & 2 \end{pmatrix} .$$

3 March 6: Exercise No. 2(b)

- **Task:** For the following 4×4 matrix M find

  (a) the rank of the matrix
  (b) a non-redundant set of linear equations in 4 variables that characterizes the linear relations among the rows of the matrix.

- **Note:** As explained in the previous class, this is essentially the same problem as that of finding linear equations for the image of the linear map

$$f_M(x) \;=\; Mx \quad .$$

- The matrix:

$$\begin{pmatrix} 1 & 2 & -4 & 7 \\ -2 & -1 & -1 & -8 \\ 5 & 7 & -11 & 29 \\ -3 & -6 & 12 & -21 \end{pmatrix}$$

- The RREF of its **transpose**:

$$\begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The rank of M is 2.

- A non-redundant characterizing set of row relations:

$$\begin{cases} -3y_1 + y_2 + y_3 &= 0 \\ 3y_1 + y_4 &= 0 \end{cases}$$

4 Matrix of a Linear Map for a Pair of Bases

The transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f]{} & \mathbf{R}^m \end{array}$$

The linear map f between Euclidean spaces has a matrix M

$$f(x) \;=\; f_M(x) \;=\; Mx$$

Definition. M is called the *matrix of ϕ for the pair of bases*

$$\mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \;\text{ and }\; \mathbf{w} \;=\; (w_1 w_2 \ldots w_m) \quad .$$

5 Linear Maps with Prescribed Values

Corollary. Given vector spaces V and W, given bases $\mathbf{v}$ in V and $\mathbf{w}$ in W, and given a matrix M of size $\dim(W) \times \dim(V)$, there is a unique linear map

$$V \xrightarrow{\phi} W$$

for which M is the matrix with respect to the given pair of bases.

Proof. Construct ϕ using the transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f_M]{} & \mathbf{R}^m \end{array}$$

Formula:

$$\phi \;=\; \alpha_{\mathbf{w}} \circ f_M \circ \alpha_{\mathbf{v}}^{-1}$$

6 Matrix of $\pi/2$ rotation in $\mathbf{R}^2$

- **Question.** What is the matrix for the rotation counterclockwise ρ by the angle with measure $\pi/2$ (radians)?
- **Note.** A rotation of $\mathbf{R}^2$ about the origin is a linear map because, as a rigid motion of the plane, it carries parallelograms to parallelograms, and addition of points in $\mathbf{R}^2$ follows the "parallelogram law".
- **Observations.**
$$\rho(1,0) \;=\; (0,1) \;\text{ and }\; \rho(0,1) \;=\; (-1,0)$$
- If J is the matrix of ρ, then
$$J_1 \;=\; (0,1) \;\text{ and }\; J_2 \;=\; (-1,0) \quad .$$
- $$J \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- J is the matrix of ρ with respect to the basis pair
$$\mathbf{v} \;=\; \mathbf{w} \;=\; \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad .$$

7 Matrix for a $\pi/2$ Rotation in $\mathbf{R}^3$

- **Question.** If P is the plane in $\mathbf{R}^3$ that is the linear span of the vectors
$$v_1 \;=\; \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \;,\; v_2 \;=\; \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \;,$$
and ρ is the rotation in space through $\pi/2$ about the axis through the origin that is perpendicular to P, specify a basis $\mathbf{v} \;=\; (v_1 v_2 v_3)$ of $\mathbf{R}^3$ relative to which the matrix of ρ is relatively simple.
- **Answer.** There is slight ambiguity since it is not possible to distinguish between clockwise and counterclockwise.

  (v_1, v_2) is a basis of the plane P

  One computes the "dot product":
$$v_1 \cdot v_2 \;=\; 1 \cdot 2 + 2 \cdot 1 + (2)(-2) \;=\; 0$$

  So v_1 and v_2 are perpendicular.

  One of the two possible rotations ρ through $\pi/2$ will satisfy:
$$\rho(v_1) \;=\; v_2 \;\text{ and }\; \rho(v_2) \;=\; -v_1 \quad .$$

  The "cross product" $v_1 \times v_2$ lies on the axis of rotation:
$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Take $v_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, a vector on the axis, as a third basis vector for $\mathbf{R}^3$.

$$\rho(v_3) = v_3$$

With $\mathbf{v} = (v_1 v_2 v_3) = \mathbf{w}$ as selected pairs of bases, the matrix of ρ is:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

8 Standard Matrix for the $\pi/2$ Rotation

- We have:

$$\mathbf{v} = (v_1 v_2 v_3) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} = Q$$

$$\rho(\mathbf{v}) = (\rho(v_1)\rho(v_2)\rho(v_3)) = (v_1 v_2 v_3) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{v}K$$

- **Note:** The second 3×3 matrix K is the matrix of the linear map ρ with respect to the basis pair $\mathbf{w} = \mathbf{v}$.

  The first 3×3 matrix Q is not the matrix of a linear map but, rather, a matrix whose columns are the standard coordinates — coordinates with respect to the standard basis — of the members of the basis $\mathbf{v}$.

  The matrix corresponding in a similar way to the standard basis $\mathbf{e} = (e_1 e_2 e_3)$ is the identity matrix, and it would be more precise, instead of writing $\mathbf{v} = Q$ to use Q to relate the row of vectors $\mathbf{v}$ to the row of vectors $\mathbf{e}$:

$$\mathbf{v} = \mathbf{e}Q \quad .$$

  Q is the *matrix for change of basis* between the basis $\mathbf{v}$ and the standard basis $\mathbf{e}$.

- For the standard matrix M of ρ one has $\rho(e_j) = Me_j$ or

$$\rho(\mathbf{e}) = (\rho(e_1)\rho(e_2)\rho(e_3)) = (e_1 e_2 e_3)M = \mathbf{e}M \quad .$$

- Since ρ is linear, and $\mathbf{v} = \mathbf{e}Q$, one has

$$\rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q\,,$$

  and, therefore, combining the various formulas:

$$\mathbf{e}QK = \mathbf{v}K = \rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q = \mathbf{e}MQ$$

  yielding the following relation among ordinary 3×3 matrices:

$$QK = MQ \text{ or } M = QKQ^{-1}$$

- Because this particular matrix Q consists of mutually perpendicular columns, all of the same length, it is particularly easy to invert:

$$Q^{-1} = (1/9)\,{}^tQ = (1/9)Q = (1/9) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \quad .$$

- $$M = \frac{1}{9}\begin{pmatrix} 4 & -1 & 8 \\ -7 & 4 & 4 \\ -4 & -8 & 1 \end{pmatrix} .$$

- M is the standard matrix of one of the two rotations through the angle $\pi/2$ about the line through the origin and the point $(2, -2, 1)$.

9 March 11: Exercise No. 1

Let g be the linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$ that is defined by $g(x) = Bx$ where B is the matrix

$$\begin{pmatrix} 1 & 2 & -4 & 3 \\ -2 & -1 & -1 & 5 \\ 1 & 3 & 2 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} .$$

Find a 4×4 matrix C for which the linear map h given by multiplication by C has the property that both $h(g(x)) = x$ and $g(h(y)) = y$ for all x and all y in $\mathbf{R}^4$.

- h is the inverse map to g. It is the linear map given by the **inverse matrix**.
- The inverse matrix: $$\begin{pmatrix} 8/3 & -29/9 & -2/9 & -71/9 \\ -1 & 4/3 & 1/3 & 10/3 \\ 2/3 & -8/9 & 1/9 & -23/9 \\ 1 & -1 & 0 & -3 \end{pmatrix}$$

10 March 11: Exercise No. 2

Let f be a linear map from $\mathbf{R}^3$ to $\mathbf{R}^3$ for which

1. $f(1, 0, 0) = (1, 2, 3)$.
2. $f(0, 1/2, 0) = (3, 2, 1)$.
3. $f(-1, 0, 2) = (4, -6, 2)$.

Find all possible 3×3 matrices A for which the formula $f(x) = Ax$ is valid for all x in $\mathbf{R}^3$.

Hint: Use the rules for abstract linearity to work out what happens under f to $(0, 1, 0)$ and $(0, 0, 1)$.

- f is determined by its values on the members of a basis.
- $\{(1, 0, 0), (0, 1/2, 0), (-1, 0, 2)\}$ is a set of 3 linearly independent vectors in $\mathbf{R}^3$, hence, a basis of $\mathbf{R}^3$.
- The columns of A are the values of f on the standard basis.
- $f(0, 1, 0) = 2f(0, 1/2, 0) = (6, 4, 2)$.
- $(0, 0, 1) = (1/2)\,((-1, 0, 2) + (1, 0, 0))$.
- $f(0, 0, 1) = (1/2)\,((4, -6, 2) + (1, 2, 3)) = (5/2, -2, 5/2)$.
- The unique matrix A is $$\begin{pmatrix} 1 & 6 & 5/2 \\ 2 & 4 & -2 \\ 3 & 2 & 5/2 \end{pmatrix} .$$

11 March 11: Exercise No. 3

For a given real number θ find a 2×2 matrix R_θ for which the linear function ρ defined by $\rho(x) = R_\theta x$ is the counterclockwise rotation of the plane through the angle of (radian) measure θ.

Hint: First work out the four special cases where θ takes the values 0, $\pi/2$, π, and $3\pi/2$.

- $$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

12 March 11: Exercise No. 4

Find a 3×3 matrix S for which the linear function σ given by $\sigma(x) = Sx$ is the reflection of $\mathbf{R}^3$ in the xz plane (where the 2^{nd} coordinate $y = 0$).

- Points in the xz plane do not move.
- Points on the y-axis are "flipped", i.e., $(0, y, 0) \mapsto (0, -y, 0)$.
- $$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$