Math 214
Follow-Up Assignment

May 13, 2002

1. Find the gradient vector of the function f that is defined by

$$f(x,y,z) \;=\; x^4y^2 - y^3z^2 + zx \quad .$$

Response:

$$(\nabla f)(x,y,z) \;=\; \left(4x^3y^2 + z,\; 2x^4y - 3y^2z^2,\; -2y^3z + x\right) \quad .$$

2. Find the divergence of the vector field F that is defined by

$$F(x,y,z) \;=\; (z^3x,\; x^3y,\; y^3z) \quad .$$

Response:

$$\frac{\partial}{\partial x}(z^3x) + \frac{\partial}{\partial y}(x^3y) + \frac{\partial}{\partial z}(y^3z) \;=\; z^3 + x^3 + y^3 \quad .$$

3. Find the double integral of the function $f(x,y) \;=\; xy^2$ on the rectangle, having sides parallel to the coordinate axes, with diagonally opposite vertices at the points $(-1,1)$ and $(5,3)$.

Response: The rectangle, say E, is described by the inequalities

$$\begin{cases} -1 \;\le\; x \;\le\; 5 \\ \;\;\;1 \;\le\; y \;\le\; 3 \end{cases} \quad .$$

Thus,

$$\begin{aligned}
\int\!\!\int_E xy^2\,dA \;&=\; \int_{-1}^{5}\int_{1}^{3} xy^2\,dy\,dx \\
&=\; \int_{-1}^{5} x\left(\int_{1}^{3} y^2\,dy\right)dx \\
&=\; \int_{-1}^{5} x\cdot\left[\frac{y^3}{3}\right]_{y=1}^{y\,=\,3} dx \\
&=\; \int_{-1}^{5} x\cdot\left(9 - \frac{1}{3}\right)dx \\
&=\; \frac{26}{3}\int_{-1}^{5} x\,dx \\
&=\; \frac{26}{3}\left[\frac{x^2}{2}\right]_{x=-1}^{x\,=\,5} \\
&=\; \frac{26}{3}\left(\frac{5^2}{2} - \frac{(-1)^2}{2}\right) \\
&=\; 104\ .
\end{aligned}$$

4. Find the tangent plane at the point $(3,-1,2)$ to the surface

$$xy^2z \;=\; 6 \quad .$$

Response: The chief task here is to find a vector normal to the tangent plane to be used as coefficient vector in the equation for the plane. Since the surface is a level set of the function

$$f(x,y,z) \;=\; xy^2z\ ,$$

the gradient ∇f of f at the given point must be normal to the tangent plane at that point (unless it vanishes).

$$\begin{aligned}(\nabla f)(x,y,z) &= (y^2 z, 2xyz, xy^2)\\ (\nabla f)(3,-1,2) &= (2,-12,3)\end{aligned}$$

Hence, the tangent plane has an equation of the form

$$2x - 12y + 3z = \text{constant} .$$

Using the fact that $(3,-1,2)$ must satisfy this equation, one obtains the equation

$$2x - 12y + 3z = 24 \quad .$$

5. Find the arc length of the helix that is given parametrically by

$$\begin{cases} x = 4\sin t \\ y = 4\cos t \\ z = 3t \end{cases}$$

for $0 \le t \le \pi/2$.

Response: This involves a straightforward application of the definition of the length of a parameterized path as the integral of the length of the derivative of a point on the path with respect to the the parameter:

$$\begin{aligned}\int_C ds &= \int_C ||(dx,dy,dz)||\\ &= \int_0^{\pi/2} \left|\left|\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)\right|\right| dt\\ &= \int_0^{\pi/2} ||(4\cos t, -4\sin t, 3)||\, dt\\ &= \int_0^{\pi/2} \sqrt{4^2\left(\cos^2 t + \sin^2 t\right) + 3^2} dt\\ &= \int_0^{\pi/2} 5dt\\ &= \frac{5\pi}{2} \ .\end{aligned}$$

6. Find the path integral of the vector field F that is defined by

$$F(x,y,z) = (x-2z,\ y+z,\ 1-y)$$

over the path given by

$$R(t) = (t^3,\ t,\ t^2)$$

for $0 \le t \le 2$.

Response: This involves a straightforward application of the definition of the integral over a parameterized path of a vector field:

$$\begin{aligned}
\int_C F dR &= \int_0^2 F(R(t)) \cdot R'(t) dt \\
&= \int_0^2 \left(t^3 - 2t^2,\, t + t^2,\, 1 - t\right) \cdot \left(3t^2,\, 1,\, 2t\right)\, dt \\
&= \int_0^2 \left((3t^5 - 6t^4) + (t^2 + t) + (-2t^2 + 2t)\right)\, dt \\
&= \int_0^2 (3t^5 - 6t^4 - t^2 + 3t)\, dt \\
&= \left[\frac{t^6}{2} - \frac{6t^5}{5} - \frac{t^3}{3} + \frac{3t^2}{2}\right]_{t=0}^{t\,=\,2} \\
&= 32 - \frac{192}{5} - \frac{8}{3} + 6 \\
&= -\frac{46}{15}\,.
\end{aligned}$$

7. Find the equation of the plane containing the point $(-1, 3, 2)$ that is normal to the line defined by the two equations

$$\left\{ \begin{array}{rcl} 3x - 4y + 2z &=& 7 \\ 9x - 5y - 3z &=& 2 \end{array} \right. .$$

Response: A key fact for this problem is that a single linear equation in 3 variables is the equation of a plane in space, and the coefficient vector of such an equation is a (unique up to a scalar multiple) normal vector to that plane.

Since a point in the required plane is given, one only needs to find a vector normal to the plane, which is the same thing as a vector parallel to the given line. The line is given as the intersection of two planes, and any normal to either plane must be normal to a vector parallel to their line of intersection. Thus a vector parallel to the line may be obtained as the "cross product" of normals to the two different planes, provided that vector is not 0 — which happens when and only when the two planes are parallel and, therefore, do not determine a line of intersection.

One finds:

$$(3, -4, 2) \times (9, -5, -3) \;=\; (22, 27, 21) \quad .$$

Hence, the required plane has an equation of the form

$$22x + 27y + 21z \;=\; \text{constant}\ ,$$

and the constant is determined by the fact that the given point must satisfy the equation:

$$22 \cdot -1 + 27 \cdot 3 + 21 \cdot 2 \;=\; -22 + 81 + 42 \;=\; 101 \quad .$$

Hence, the required plane has the equation:

$$22x + 27y + 21z \;=\; 101 \quad .$$

8. Find the centroid of the solid cone

$$x^2 + y^2 \leq z^2\,, \quad 0 \leq z \leq 2 \quad .$$

Response: The lateral boundary of the solid cone, given by $x^2 + y^2 = z^2$, is the surface obtained by rotating the line $z = y$ in the plane $x = 0$ about the z-axis. Thus, for $z \geq 0$ the section of the solid cone by the plane normal to the z-axis through the point $(0, 0, z)$ is the disk of radius z with center $(0, 0, z)$. The radius of this (inverted) cone's base, as well as its altitude, is 2, and, therefore its volume is

$$\frac{1}{3}\pi a^2 h \;=\; \frac{8\pi}{3} \quad .$$

By symmetry its centroid lies on the z-axis, and the only issue is what is its z-coordinate:

$$\begin{aligned}
\bar{z} &= \frac{1}{\text{volume}} \int\int\int z\,dV \\
&= \frac{3}{8\pi} \int_0^2 zdz \int\int_{x^2+y^2 \le z^2} dxdy \\
&= \frac{3}{8\pi} \int_0^2 z \cdot (\text{Area of disk of radius } z)\,dz \\
&= \frac{3}{8\pi} \int_0^2 \pi z^3 dz \\
&= \frac{3}{8} \cdot 4 \\
&= \frac{3}{2}
\end{aligned}$$

Hence, the centroid of the cone is the point $(0, 0, 3\pi/2)$.

9. Find the surface integral of the vector field F over the sphere, oriented by its outer normal, of radius 3 with center at the origin when F is given by

$$F(x, y, z) \;=\; (x^3,\; y^3,\; z^3) \quad .$$

Response: The vector field is well-behaved everywhere, particularly in the ball surrounded by the given sphere. The divergence theorem may be applied.

$$(\text{div}F)(x, y, z) \;=\; (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot F \;=\; \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) \;=\; 3(x^2 + y^2 + z^2) \quad .$$

Therefore,

$$\int\int_S F \cdot N d\sigma \;=\; \int\int\int_B \text{div}F\,dV \quad .$$

In polar coordinates, observing that $dV = \rho^2 \sin\phi\,d\rho d\phi d\theta$ and that $\text{div}F = 3\rho^2$, the triple integral with radius $a = 3$ becomes:

$$3 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi d\phi \int_0^3 \rho^4\,d\rho \;,$$

which evaluated becomes

$$\frac{12\pi}{5} a^5 \;=\; \frac{2916\pi}{5} \quad .$$

10. For a conservative vector field in the plane what can be said about its integral over the circle, traversed counterclockwise, of radius 2 with center at the point $(3, 4)$? Explain your answer.

Response: Because *conservative* in this context means that the vector field is the gradient of a scalar, one may use the formula

$$\int_C \nabla f \;=\; f(B) - f(A)$$

where A and B are, respectively, the initial and final points of an oriented curve C. A trip around a circle, no matter where it begins, will end where it begins. Thus, $A = B$, and the integral of a vector field that is conservative in the plane around *any* circle must be zero.