Affine 3-folds and tetrahedra in 4-space

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Definition. An *affine r-fold in* $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$ of the form

$$a + t_1 v_1 + t_2 v_2 + \ldots + t_r v_r \quad \text{with } t_1, t_2, \ldots, t_r \text{ varying in } \mathbf{R} \tag{1}$$

where a is a given point in $\mathbf{R}^n$ and $v_1, v_2, \ldots, v_n$ are given vectors in $\mathbf{R}^n$ subject to the condition that none of the vectors v_j may be omitted without changing the set (1).

The following data determine a plane, i.e., an affine 2-fold, in $\mathbf{R}^4$:

Example 1.

$$a \;=\; (1,-2,0,1)\,; \quad v_1 \;=\; (1,2,1,3)\,,\; v_2 \;=\; (2,-1,1,0) \tag{2}$$

Moreover, if one takes

$$b \;=\; a + v_1 \;=\; (2,0,1,4) \text{ and } c \;=\; a + v_2 \;=\; (3,-3,1,1),$$

then a, b, and c are the vertices of triangle $T = abc$ in $\mathbf{R}^4$, which is the set of all points

Example 2.

$$(1,-2,0,1) + t(1,2,1,3) + u(2,-1,1,0) \quad \textit{with } t,u \geq 0 \textit{ and } t+u \leq 1 \quad .$$

Equivalently, T is the set of all points

$$s(1,-2,0,1) + t(2,0,1,4) + u(3,-3,1,1) \quad \text{with } s,t,u \geq 0 \text{ and } s+t+u \;=\; 1 \quad .$$

What is the area of T? The idea is that the plane (2) is a normal Euclidean plane where lengths and angles may be computed using the dot product in $\mathbf{R}^4$. Thus the area of the triangle is half the product of the length of a base and the length of the altitude drawn to that base. If we take the side ab as base, then its length is $\|b-a\| \;=\; \|v_1\| \;=\; \sqrt{15}$. For the altitude drawn from c to ab, one observes that in the decomposition of the vector

$$\vec{ac} \;=\; v_2$$

as a sum of components parallel and perpendicular to v_1 the perpendicular component lies over the altitude drawn from c to ab. One finds

$$\begin{aligned} \mathrm{proj}_{v_1}(v_2) \;&=\; \frac{1}{15}(1,2,1,3) \\ \mathrm{perp}_{v_1}(v_2) \;&=\; \frac{1}{15}(29,-17,14,-3) \end{aligned} \tag{3}$$

So the area of T is given by

$$A \;=\; \frac{1}{2}Bh \;=\; \frac{1}{2}\sqrt{15}\left(\frac{\sqrt{1335}}{15}\right) \;=\; \frac{1}{2}\sqrt{89} \quad .$$

If, in addition to the point a and the vectors v_1, v_2, one brings another vector $v_3 = (-1,3,2,-1)$ into the picture, then one has data determining an affine 3-fold in $\mathbf{R}^4$:

Example 3.

$$a \;=\; (1,-2,0,1)\,; \quad v_1 \;=\; (1,2,1,3)\,,\; v_2 \;=\; (2,-1,1,0)\,,\; v_3 \;=\; (-1,3,2,-1) \tag{4}$$

Just as an affine 2-fold in $\mathbf{R}^3$, i.e., a plane, may be described alternatively as the set of all points satisfying a single linear equation, an affine 3-fold in $\mathbf{R}^4$ — and, more generally, an affine $(n-1)$-fold in $\mathbf{R}^n$ — may be described as the set of all points satisfying a single linear equation in which the vector of coefficients of the coordinates is a vector that is perpendicular to the 3-fold. To find the equation for the present example one begins by looking for a vector u that is perpendicular to each of the vectors v_1, v_2, and v_3. The relations $u \cdot v_1 = 0$, $u \cdot v_2 = 0$, and $u \cdot v_3 = 0$ amount to 3 equations for the coefficients of u and to make u specific one may add the equation $u_4 = 1$. Thus,

$$\begin{aligned} u_1 + 2u_2 + u_3 + 3u_4 &= 0 \\ 2u_1 - u_2 + u_3 &= 0 \\ -u_1 + 3u_2 + 2u_3 - u_4 &= 0 \\ u_4 &= 1 \end{aligned}$$

Solution of this system of 4 equations yields:

$$u = (-9/5, -8/5, 2, 1) ,$$

and to eliminate denominators without changing the direction of this vector, one may replace it with its scalar multiple by -5:

$$u = (9, 8, -10, -5) \quad .$$

Therefore, this affine 3-fold has equation of the form

$$9x_1 + 8x_2 - 10x_3 - 5x_4 = \text{constant} ,$$

and the constant is determined by evaluating the left side at the point a. Thus the equation of this "hyperplane" is:

$$9x_1 + 8x_2 - 10x_3 - 5x_4 = -12 \quad .$$

The affine 3-fold contains, in particular, the following 4 points:

$$a = (1, -2, 0, 1) , \; b = a + v_1 = (2, 0, 1, 4) , \; c = a + v_2 = (3, -3, 1, 1) , \; d = a + v_3 = (0, 1, 2, 0) ,$$

and these four points are the vertices of the tetrahedron $abcd$ that sits inside a copy of $\mathbf{R}^3$ that itself is a hyperplane in $\mathbf{R}^4$. What is the volume of this tetrahedron?

If we take the triangle abc as "base" with area $A = (1/2)\sqrt{89}$, as previously calculated, principles from school geometry indicate the volume of the tetrahedron should be given by the formula

$$V = \frac{1}{3} A h \tag{5}$$

where h is the "altitude" drawn from the vertex d to the base. The idea for finding the altitude is to decompose the vector $v_3 = \vec{ad}$ into the sum $w' + w''$ of two components, with w' in the base and w'' perpendicular to the base. Thus,

$$w' = tv_1 + uv_2 \quad \text{and} \quad w'' = v_3 - tv_1 - uv_2 ,$$

while the condition $w'' \perp \Delta abc$ gives the two conditions $w'' \cdot v_1 = 0$ and $w'' \cdot v_2 = 0$ which amount to a pair of linear equations for the two scalars t and u. One finds $t = 27/89$ and $u = -49/89$ with the result that

$$w' = (1/89)(-71, 103, -22, 81) \quad w'' = (1/89)(-18, 164, 200, -170) ,$$

and, therefore,

$$h = \|w''\| = \frac{6\sqrt{30}}{\sqrt{89}} \quad .$$

Applying the formula (5), one sees that the volume of the tetrahedron $abcd$ is $\sqrt{30}$.