Math 220 **Exam 3 Solutions** **Spring '01**

1. Let $A = \begin{bmatrix} 68 & 3 & -51 & 39 \\ -72 & -4 & 54 & -42 \\ 147 & 6 & -112 & 87 \\ 81 & 3 & -63 & 50 \end{bmatrix}$.

a) What is the characteristic polynomial $\mathrm{ch}_A(\lambda)$?

SOLUTION: We use the Maple command

factor(charpoly(A,lambda));

obtaining

$$\mathrm{ch}_A(\lambda) = (\lambda+1)^2(\lambda-2)^2.$$

b) For each eigenvalue of A, find a basis for the associated eigenspace.

SOLUTION: The eigenvalues are -1 and 2.

The eigenspace of -1 is the nullspace of $A+I$. Applying the Maple command "nullspace" to the matrix $A+I$ (which may be input by copying and pasting A, and manually adding I to it), we see that $\mathrm{N}(A+I)$ has basis $\mathbf{v}_1, \mathbf{v}_2$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 10 \\ 0 \\ 1 \end{bmatrix}$.

The eigenspace of 2 is the nullspace of $A-2I$. Using the same methods, Maple shows that $\mathrm{N}(A-2I)$ has basis $\mathbf{v}_3, \mathbf{v}_4$, where $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ -\frac{5}{3} \end{bmatrix}$.

c) Is A diagonalizable? If so, find a matrix P such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

SOLUTION: We set $P = [\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3 \,|\, \mathbf{v}_4] = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -6 & 10 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & -\frac{5}{3} \end{bmatrix}$.

Since the columns of P form a basis of $\mathbf{R}^4$ consisting of eigenvectors of A, $P^{-1}AP$ is a diagonal matrix whose diagonal entries are the eigenvalues associated (in order) with the columns of P: $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. This can be verified via the Maple command "multiply(inverse(P),A,P);".

2. Let $A = \begin{bmatrix} -17 & 0 & 10 & -5 \\ 22 & 3 & -13 & 9 \\ -45 & 0 & 28 & -15 \\ -30 & 0 & 20 & -12 \end{bmatrix}$.

   a) What is the characteristic polynomial $\mathrm{ch}_A(\lambda)$?

   SOLUTION: Using the same methods as in problem 1, we obtain

   $$\mathrm{ch}_A(\lambda) = (\lambda+2)^2(\lambda-3)^2.$$

   Thus, the eigenvalues are -2 and 3.

   b) For each eigenvalue of A, find a basis for the associated eigenspace.

   SOLUTION: The eigenspace for -2 is the nullspace of $A+2I$. Using Maple as in problem 1, we see $\mathrm{N}(A+2I)$ has basis $\mathbf{v}_1, \mathbf{v}_2$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

   The eigenspace for 3 is the nullspace for $A-3I$. This time, we find that the nullspace is one-dimensional, with basis $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

   c) Is A diagonalizable? If so, find a matrix P such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

   SOLUTION: The algebraic multiplicity of the eigenvalue 3 is two, but its geometric multiplicity (the dimension of its eigenspace) is one. Since these are different, A can't be diagonalizable.

3. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 7x_2 + 4x_3 \\ 5x_1 + 4x_2 - 2x_3 \\ x_1 - 2x_2 + 4x_3 \end{bmatrix}.$$

a) What is the matrix for T with respect to the standard basis $\mathcal{E} = \{e_1, e_2, e_3\}$ of $\mathbf{R}^3$?

SOLUTION: We can read off the matrix of T with respect to the standard basis from the coefficients of the equation for T:

$$[T]_{\mathcal{E}} = \begin{bmatrix} 3 & -7 & 4 \\ 5 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

b) Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$. What is the matrix $[T]_{\mathcal{B}}$ of T with respect to $\mathcal{B}$?

SOLUTION:

$$\begin{aligned} [T]_{\mathcal{B}} &= [I]_{\mathcal{B}\mathcal{E}}[T]_{\mathcal{E}}[I]_{\mathcal{E}\mathcal{B}} \\ &= P^{-1}[T]_{\mathcal{E}}P, \end{aligned}$$

where

$$P = [\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 6 \end{bmatrix}$$

Since we computed $[T]_{\mathcal{E}}$ in part a), we can find $P^{-1}[T]_{\mathcal{E}}P$ using Maple. We get

$$[T]_{\mathcal{B}} = \begin{bmatrix} 4 & -77 & 109 \\ 1 & 50 & -64 \\ 0 & 33 & -43 \end{bmatrix}$$

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4. Let $T : P_2 \to P_2$ be given by $T(p) = p(7 - 5x)$.

a) What is the matrix of T with respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$?

SOLUTION: To find $[T]_{\mathcal{E}}$, we'll need to find $T(1)$, $T(x)$ and $T(x^2)$. For a polynomial p, $T(p)$ is what we get if we substitute $7 - 5x$ in place of x in the formula for p. Thus, $T(1) = 1$ (x doesn't appear at all in the formula for the constant polynomial 1). We also have $T(x) = 7 - 5x$, and $T(x^2) = (7 - 5x)^2 = 49 - 70x + 25x^2$. By definition,

$$[T]_{\mathcal{E}} = \big[[T(1)]_{\mathcal{E}} \,|\, [T(x)]_{\mathcal{E}} \,|\, [T(x^2)]_{\mathcal{E}}\big] = \begin{bmatrix} 1 & 7 & 49 \\ 0 & -5 & -70 \\ 0 & 0 & 25 \end{bmatrix}.$$

b) What is $\det(T)$?

SOLUTION: $\det(T) = \det[T]_{\mathcal{E}}$. Since $[T]_{\mathcal{E}}$ is upper triangular, its determinant is the product of its diagonal entries: $\det(T) = -125$.

c) What is the trace of T?

SOLUTION: $\operatorname{tr}(T) = \operatorname{tr}([T]_{\mathcal{E}})$. The trace of any matrix is the sum of its diagonal entries, so $\operatorname{tr}(T) = 21$.

d) What is the characteristic polynomial $\operatorname{ch}_T(\lambda)$?

SOLUTION:

$$\operatorname{ch}_T(\lambda) = \operatorname{ch}_{[T]_{\mathcal{E}}}(\lambda) = \det(\lambda I - [T]_{\mathcal{E}}) = \begin{vmatrix} (\lambda - 1) & -7 & -49 \\ 0 & (\lambda + 5) & 70 \\ 0 & 0 & (\lambda - 25) \end{vmatrix}.$$

Since this matrix is upper triangular, its determinant is the product of its diagonal entries, so

$$\operatorname{ch}_T(\lambda) = (\lambda - 1)(\lambda + 5)(\lambda - 25).$$

Thus, the eigenvalues of T are 1, -5 and 25.

e) Give a basis for each eigenspace of T.

SOLUTION: We use Maple to find bases for the eigenspaces of $A = [T]_{\mathcal{E}}$, and then find polynomials whose coordinates with respect to $\mathcal{E}$ are these basis vectors.

For the eigenvalue 1, the eigenspace of A has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This corresponds to the constant polynomial 1, so the eigenspace of T with respect to 1 has basis $\mathbf{p}_1 = 1$.

For the eigenvalue -5, the eigenspace of A has basis $\begin{bmatrix} -\frac{7}{6} \\ 1 \\ 0 \end{bmatrix}$,
which corresponds to the polynomial $-\frac{7}{6}+x$. Thus, the eigenspace for T with respect to -5 has basis $\mathbf{p}_2 = -\frac{7}{6}+x$. (We could also use $7-6x$, a multiple of this $\mathbf{p}_2$.)

For the eigenvalue 25, the eigenspace of A has basis $\begin{bmatrix} \frac{49}{36} \\ -\frac{7}{3} \\ 1 \end{bmatrix}$.
This corresponds to the polynomial $\mathbf{p}_3 = \frac{49}{36} - \frac{7}{3}x + x^2$. (As above, we could replace this with $49 - 84x + 36x^2$.)

Note: You can check directly that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are eigenvectors for the stated eigenvalues. For $\mathbf{p}_1$, this is given by our calculation of $T(1)$. In the other cases, you can use Maple's "expand" command for polynomial arithmetic. E.g., for $\mathbf{p}_2$, use

$$\text{expand}\left(-\frac{7}{6} + (7 - 5*x) + 5*(-\frac{7}{6}+x)\right);$$

This computes $T(\mathbf{p}_2) + 5\mathbf{p}_2$, and the output of 0 verifies that $\mathbf{p}_2$ is an eigenvector for T with eigenvalue -5.

f) Is T diagonalizable? If so, find a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal, and display the diagonal matrix $[T]_{\mathcal{B}}$.

SOLUTION: T is diagonalizable because $\mathrm{ch}_T(\lambda)$ factors completely over $\mathbf{R}$ and each eigenvalue has algebraic multiplicity 1. Any basis $\mathcal{B}$ consisting of eigenvectors will have the property that $[T]_{\mathcal{B}}$ is diagonal, with the diagonal entries being the eigenvalues associated to the basis elements (in the order they appear in the basis). Thus, take $\mathcal{B} = \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, which gives

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$