Math 220 **Exam 1 Solutions** **Fall '01**

1. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 1 \\ 7 & -2 & 1 \end{bmatrix}$

a) Find A^{-1}.

SOLUTION: We augment A by the identity matrix and begin by using the first row to simplify the other entries in the first column:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 & 1 & 0 \\ 7 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 1 & -2 & -2 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 4 & 5 & 7 & 0 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 4 & 5 & 7 & 0 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 4 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & -5 & 2 \\ 0 & 0 & 1 & -1 & 4 & -2 \end{array}\right]$$

Since the left hand side is now I, $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -5 & 2 \\ -1 & 4 & -2 \end{bmatrix}$.

b) Use A^{-1} to find the solution of $Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

SOLUTION: $x = A^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 - 2b_2 + b_3 \\ 3b_1 - 5b_2 + 2b_3 \\ -b_1 + 4b_2 - 2b_3 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -4 & -8 & -2 \\ -9 & -18 & -8 \\ 9 & 18 & 3 \end{bmatrix}$.

a) For which values of b_1, b_2, b_3 is the system $Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ consistent?

SOLUTION: We do Gauss elimination on A augmented by b:

$$\left[\begin{array}{ccc|c} -4 & -8 & -2 & b_1 \\ -9 & -18 & -8 & b_2 \\ 9 & 18 & 3 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} -4 & -8 & -2 & b_1 \\ -9 & -18 & -8 & b_2 \\ 0 & 0 & -5 & b_2 + b_3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} -4 & -8 & -2 & b_1 \\ -1 & -2 & -4 & -2b_1 + b_2 \\ 0 & 0 & -5 & b_2 + b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & -2 & -4 & -2b_1 + b_2 \\ -4 & -8 & -2 & b_1 \\ 0 & 0 & -5 & b_2 + b_3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} -1 & -2 & -4 & -2b_1 + b_2 \\ 0 & 0 & 14 & 9b_1 - 4b_2 \\ 0 & 0 & -5 & b_2 + b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & -2 & -4 & -2b_1 + b_2 \\ 0 & 0 & -1 & 9b_1 - b_2 + 3b_3 \\ 0 & 0 & -5 & b_2 + b_3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} -1 & -2 & 0 & -38b_1 + 5b_2 - 12b_3 \\ 0 & 0 & -1 & 9b_1 - b_2 + 3b_3 \\ 0 & 0 & 0 & -45b_1 + 6b_2 - 14b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 38b_1 - 5b_2 + 12b_3 \\ 0 & 0 & 1 & -9b_1 + b_2 - 3b_3 \\ 0 & 0 & 0 & -45b_1 + 6b_2 - 14b_3 \end{array}\right].$$

The reduction of A has one row of zeros. We see that the system is consistent when $-45b_1 + 6b_2 - 14b_3 = 0$.

b) Find the general solution whenever the system is consistent.

SOLUTION: The variable x_2 is the only non-pivot, so we set $x_2 = t$ and solve for the pivots.

The first row of the reduction gives $x_1 + 2x_2 = 38b_1 - 5b_2 + 12b_3$, so $x_1 = 38b_1 - 5b_2 + 12b_3 - 2t$.

The second row of the reduction gives $x_3 = -9b_1 + b_2 - 3b_3$. Putting these together, we get

$$x = \begin{bmatrix} 38b_1 - 5b_2 + 12b_3 - 2t \\ t \\ -9b_1 + b_2 - 3b_3 \end{bmatrix} = \begin{bmatrix} 38b_1 - 5b_2 + 12b_3 \\ 0 \\ -9b_1 + b_2 - 3b_3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 3 & 6 & 12 \\ -6 & -17 & -34 \\ 4 & 10 & 19 \end{bmatrix}$.

Recall that that $Ax = 3x$ can be rewritten as $(A - 3I)x = 0$. Use this to solve $Ax = 3x$. (Hint: there are nontrivial solutions.)

SOLUTION: We subtract $3I$ from A and do Gauss elimination:

$$\begin{bmatrix} 0 & 6 & 12 \\ -6 & -20 & -34 \\ 4 & 10 & 16 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ -6 & -20 & -34 \\ 4 & 10 & 16 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 1 & 2 \\ -6 & 0 & 6 \\ 4 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} -6 & 0 & 6 \\ 0 & 1 & 2 \\ 4 & 0 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 4 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We are solving the homogeneous system $(A - 3I)x = 0$, so we can read off the answer from the reduced matrix. The only non-pivot is x_3, so we set $x_3 = t$.

The first line gives $x_1 - x_3 = 0$, so $x_1 = t$. The second line gives $x_2 + 2x_3 = 0$, so $x_2 = -2t$. Thus

$$x = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 2 & 1 & 2 \\ 3 & 2 & 4 & 1 \\ -3 & -5 & 6 & -4 \end{bmatrix}$. Use Gauss elimination to calculate $\det A$.

SOLUTION:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 4 & 2 & 1 & 2 \\ 3 & 2 & 4 & 1 \\ -3 & -5 & 6 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -4 & 9 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -5 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & -4 & 9 & -3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & 10 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & -3 & 10 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 7 & -6 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 7 & -6 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -6 \end{vmatrix} = 5(-1)(-1)(-6) = -30,$$

as the last displayed matrix is upper triangular.