Problem 5.29 (Ferry) (A) (Borel Conjectures)

Existence: *Given a Poincaré duality group π, its $K(\pi, 1)$, K, is homotopy equivalent to a closed, topological m-manifold M.*

Remarks: Note that Problem 3.77 is a 3-dimensional version of this Conjecture.

By definition, π is a Poincaré duality group if K satisfies Poincaré duality over $\mathbb{Z}\pi$ with respect to a fundamental class in $H_m(K;\mathbb{Z}) = \mathbb{Z}$. An attempt at the Conjecture naturally breaks up into three beginning steps:

(Step 1) Prove that π is finitely presented.

If this is true, Browder [151,1972,Invent. Math.] and Brown [152,1982] show that K is dominated by a finite complex.

(Step 2) Prove that K can be chosen to be a finite complex.

The obstruction to K being homotopy equivalent to a finite complex is in $\widetilde{K_0}(\mathbb{Z}\pi)$, which vanishes if (C) below is true since $\widetilde{K_0}(\mathbb{Z}\pi)$ is a summand of $Wh(\mathbb{Z}[\pi \times \mathbb{Z}])$.

Poincaré duality gives a $\mathbb{Z}\pi$ homomorphism (cap product) between the based chain complexes C^k and C_{m-k} which is a $\mathbb{Z}\pi$ homology equivalence and thus has Whitehead torsion in $Wh(\mathbb{Z}\pi)$. K is called a *simple* PD space if this torsion is zero.

(Step 3) Prove that K is simple.

Note that all closed, compact manifolds have finitely presented π_1, are homotopy equivalent to finite complexes [585,Kirby & Siebenmann,1977], and are simple, and sometimes these properties are assumed in the Conjecture. With these properties, one is ready to apply the surgery exact sequence described below.

Uniqueness: *If $f : M^m \to N^m$ is a homotopy equivalence between closed, aspherical manifolds, then f is homotopic to a homeomorphism.*

Remarks: This is a topological analog of Mostow rigidity and is true in dimensions ≥ 5 in case M (but not necessarily N) is a non-positively curved Riemannian manifold [294,Farrell & Jones,1989,J. Amer. Math. Soc.]. Note the relation with Problem 4.83.

(B) *Under the assumptions of the Uniqueness conjecture, is f a tangential equivalence, i.e. is $f^*(T_N)$ stably isomorphic to T_M?*

Remarks: This is a version of the integral Novikov conjecture for $\pi_1(M)$.

(C) (Hsiang) *If Γ is a torsion-free, finitely presented group, then is $Wh(\mathbb{Z}\Gamma) = 0$?* .

Remarks: (C) may be a step in proving (B), for if f is homotopic to a homeomorphism, it must be homotopic to a simple homotopy equivalence which implies that

$Wh(\mathbb{Z}\pi_1(M)) = 0$. Note that M aspherical implies that $\pi_1(M)$ is torsion free (and of course finitely presented). The answer to (C) is yes when Γ is π_1 of a non-positively curved polyhedron [515, Hu, 1993, J. Differential Geom.].

The old Problem 3.32 is a special case of this conjecture which was proposed in [511, Hsiang, 1984].

(D) *Is there a closed, aspherical, ANR, homology manifold with Quinn index* $\neq 1$*?*

Remarks: Compare Problem 4.69. If we do not require the homology manifold to be aspherical, then these exist; in fact there are ones which are homotopy equivalent to S^m for each $m \geq 6$. A *yes* answer to (D) implies that either the Borel existence conjecture or the integral Novikov conjecture (see below) fails. For if H is the homology manifold asked for in (D), and M is the manifold conjectured in (A), then, using a version of the material below which encompasses homology manifolds (see [1103, Weinberger, 1994]), H and M (which live in $\mathcal{S}(X)$) go to different elements in $H_m(B\pi; \mathbb{L}_0)$ but the same element in $L^s_m(\mathbb{Z}\pi)$.

Further remarks: All these problems are versions assuming asphericity of classical problems in surgery theory. We give a very brief sketch below, but an excellent source for this material is [1103, Weinberger, 1994].

Associated to an m-manifold M^m is a map $f : M \to B\pi$ which classifies the universal cover of M ($\pi = \pi_1(M)$). Let $\alpha \in H^k(B\pi; \mathbb{Q})$. Then the rational Novikov conjecture for α is that $\langle L_{4i}(M) \cup f^*(\alpha), \mu_M \rangle$ is a homotopy invariant.

Any α provides a map $g : (B\pi \times B^l, \partial) \to (S^{k+l}, *)$ for which $\alpha = g^*(1)$, $1 \in H^{k+l}(S^{k+l}; \mathbb{Q})$. After making $g(f \times id) : (M \times B^l, \partial) \to (S^{k+l}, *)$ transverse to a point $p \neq *$, then the preimage of p is a $4i$-manifold in $M \times B^l$ whose signature equals $\langle L_{4i}(M) \cup f^*(\alpha), \mu_M \rangle$. Thus the rational Novikov conjecture asks whether the signatures of certain submanifolds of M (or $M \times B^l$) depending on elements of $H^k(B\pi; \mathbb{Q})$ are homotopy invariants.

The modern way to attack this conjecture is, however, not via the above description. Instead, recall the surgery exact sequence: if X^m is a Poincaré complex with Spivak normal fibration given by $X^m \xrightarrow{\nu} BG$, then suppose, to get started, that ν has a lift to $X^m \xrightarrow{\nu'} BTOP$. Then ν' provides a basepoint in $\mathcal{N}(X)$ (the set of liftings of $X^m \xrightarrow{\nu} BG$) and $[X, G/TOP]$ acts simply transitively on $\mathcal{N}(X)$. Then the surgery exact sequence is

$$\to L^s_{m+1}(\mathbb{Z}\pi) \to \mathcal{S}(X) \to \mathcal{N}(X) \xrightarrow{\theta} L^s_m(\mathbb{Z}\pi)$$

where, (1) an element (X, ν') of $\mathcal{N}(X)$ provides a normal map $g : N^m \to X$ (N is a manifold, g is degree one, and g is covered by a bundle map from the stable normal bundle of N to the bundle over X induced by ν'), and $\theta(X, \nu')$ is the obstruction to finding a normal bordism to a manifold M simple homotopy equivalent to X, and (2) $\mathcal{S}(X)$ is the equivalence classes of pairs (M, g), with $M \xrightarrow{g} X$ being a simple homotopy equivalence, and (M', g') equivalent to (M, g) if there exists a homeomorphism of M to M' making the obvious diagram homotopy commute.

The surgery exact sequence is now studied via the following commutative diagram:

$$\begin{array}{ccccc} & \mathcal{N}(X) & \xrightarrow{\ \theta\ } & L^s_m(\mathbb{Z}\pi) & \\ & \Big\downarrow & & \Big\uparrow {\scriptstyle A_\pi} \ \text{assembly map} & \\ \pi_m(X_+ \wedge \mathbb{L}_0) = & H_m(X;\mathbb{L}_0) & \longrightarrow & H_m(B\pi;\mathbb{L}_0) & = \ \pi_m(B\pi_+ \wedge \mathbb{L}_0) \end{array}$$

where $\mathbb{L}_0$ is the quadratic L-theory spectrum of the trivial group; thus

$$\pi_k(\mathbb{L}_0)) = \begin{cases} 0 & k < 0 \\ \mathbb{Z} & k \equiv 0 \pmod 4 \\ 0 & k \equiv 1 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z} & k \equiv 2 \pmod 4 \\ 0 & k \equiv 3 \pmod 4 \end{cases}$$

Very roughly, one can break up the surgery obstruction for $N^m \to X$ into the pieces over each simplex of X (since these simplices are simply connected, one is led to $\mathbb{L}_0$ and the simply connected surgery groups, $\mathbb{Z}$ (signature), 0, $\mathbb{Z}/2\mathbb{Z}$ (Kervaire invariant), 0); putting the pieces together leads to the assembly map A_π. (The real assembly map, at the level of spectra, is $B\pi_+ \wedge \mathbb{L}_0 \to \mathbb{L}_0(\mathbb{Z}\pi)$ where the latter spectrum is that for which $\pi_m(\mathbb{L}_0(\mathbb{Z}\pi)) = L^s_m(\mathbb{Z}\pi)$).

In this language, we have the following versions of the Novikov and Borel Conjectures (which are purely algebraic):

- **Integral Novikov Conjecture**: *If π is torsion free, then A_π is injective.*
- **Rational Novikov Conjecture:** *$A_\pi \otimes \mathbb{Q}$ is injective.*
- **Borel Conjecture for π:** *If π is a Poincaré duality group, then A_π is an isomorphism. (This is false for many other π). This Conjecture implies the Borel Existence and Uniqueness Conjectures above.*

- **Modern Borel Conjecture:** *The L–theory and K–theory assembly maps are all isomorphisms for Poincaré duality groups.*