THEORY OF BERGMAN SPACES

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$ and let dv denote volume measure on Ω. For any $p > 0$ the Bergman space $A^p = A^p(\Omega)$ is defined as the subspace of holomorphic functions in $L^p(\Omega, dv)$. In other words,

$$A^p(\Omega) = L^p(\Omega, dv) \cap H(\Omega),$$

where $H(\Omega)$ is the space of all holomorphic functions in Ω. Elementary arguments show that $A^p(\Omega)$ is closed in $L^p(\Omega, dv)$.

The space A^2 is a Hilbert space with the inner product inherited from $L^2(\Omega, dv)$. Furthermore, for any fixed $w \in \Omega$, the point evaluation $f \mapsto f(w)$ is a bounded linear functional on A^2, so it follows from the Riesz representation theorem in functional analysis that there exists a unique function $K_w \in A^2$ such that

$$f(w) = \langle f, K_w \rangle = \int_\Omega f \, \overline{K_w} \, dv$$

for all $f \in A^2$. The function K on $\Omega \times \Omega$ defined by $K(z, w) = K_w(z)$ is called the Bergman kernel of Ω.

It is probably because of S. Bergman's book "The Kernel Function and Conformal Mapping" (AMS, 1950) that prompted the use of the term "Bergman spaces", although the study of such spaces had begun much earlier. In particular, the book "Topics in the Theory of A^p_α Spaces" by A.E. Djrbashian and F.A. Shamoian (Leipzig, 1988) gives a good account of early function theory for Bergman spaces in the unit disk and contains many references on the topic that were earlier than Bergman's book.

The modern theory of Bergman spaces consists of three closely related topics.

The function theory of Bergman spaces in the unit disk. The following books discuss topics in this area:

1. *Theory of Bergman Spaces* by Hedenmalm, Korenblum, and Zhu, Springer, 2000.
2. *Bergman Spaces* by Duren and Schuster, AMS, 2004.
3. *Topics in the Theory of A^p_α Spaces* by Djrbashian and Shamoian, Leipzig, 1988.

The operator theory related to Bergman spaces. The books in this area include

1. *Operator Theory in Function Spaces* by Zhu, Marcel Dekker, 1990, and second edition, AMS, 2007.
2. *Composition Operators on Spaces of Analytic Functions* by Cowen and MacCluer, CRC Press, 1995.
3. *Composition Operators and Classical Function Theory* by Shapiro, Springer, 2003.

Bergman spaces in several complex variables. The following books contain the most important results in this area.

1. *Spaces of Holomorphic Functions in the Unit Ball* by Zhu, Springer, 2005.
2. *Function Theory in the Unit Ball of* $\mathbb{C}^n$ by Rudin, Springer, 1980.
3. *Function Theory of Several Complex Variables* by Krantz, Wiley and Sons, 1982.
4. *Holomorphic Functions and Integral Representations in Several Complex Variables* by Range, Springer, 1986.

The following paragraphs serve as an introduction to several prominent problems in these areas, several of which are still open.

2. ZERO SETS

Let X be a space of analytic functions in a planar domain Ω. A zero set (or zero sequence) for X is a sequence $\{z_n\}$ in Ω with the property that there exists a function $f \in X$ that vanishes on and only on $\{z_n\}$.

A classical example is the space H^∞ of bounded analytic functions in the unit disk $\mathbb{D}$. It is a celebrated theorem in complex analysis that a sequence $\{z_n\}$ in $\mathbb{D}$ is a zero set for H^∞ if and only if

$$\sum_n (1 - |z_n|) < \infty,$$

which is called the Blaschke condition. It turns out that the Blaschke condition also characterizes zeros sets for another family of important function spaces in complex analysis: Hardy spaces (denoted by H^p).

The problem of characterizing zero sets for the Bergman space $A^p(\mathbb{D})$ has attracted the attention of several generations of complex analysts. The problem is still open, but several partial results have been obtained by Horowitz, Korenblum, and Seip.

3. CYCLIC VECTORS

Let X be a topological vector space of analytic functions in the unit disk with the property that $pf \in X$ whenever $f \in X$ and p is a polynomial. A cyclic vector for X is a function $f \in X$ with the property that the set of polynomial multiples pf is dense in X.

Cyclic vectors for the Hardy space H^p, $0 < p < \infty$, are known to be exactly the outer functions in H^p. Roughly speaking, a function $f \in H^p$ is outer if it is uniquely determined by its boundary modulus $|f(e^{it})|$ (up to a constant multiple). In fact, an outer function can easily be recovered from its boundary modulus via an integral transform.

The problem of characterizing cyclic vectors for the Bergman space A^p is interesting to people in both complex analysis and operator theory. Although numerous classes of examples of cyclic and noncyclic vectors are known, a satisfactory characterization is still lacking. The problem will probably remain open for some years to come.

4. INVARIANT SUBSPACES

A closed subspace I of A^p is called an invariant subspace if $pf \in I$ whenever $f \in I$ and p is a polynomial. Obviously, it suffices to check the condition for the special polynomial $p(z) = z$. For this reason, invariant subspaces of A^p are also called z-invariant subspaces.

There are several outstanding problems concerning invariant subspaces of A^p in general, and A^2 in particular. These problems are important in both function theory and operator theory.

The first problem is motivated by a classical theorem of Beurling's, which states the z-invariant subspaces of the Hardy space H^p are exactly those of the form φH^p, where φ is a so-called inner function. A function $f \in H^p$ is called an inner function if its boundary modulus is 1 almost everywhere (hence its boundary modulus tells nothing about the function, as opposed to outer functions that are completely determined by their boundary modulus). Invariant subspaces of A^p are known to be much more complicated. In particular, most invariant subspaces of A^p are not of "the form φA^p" (a closure has to be taken in order to obtain a closed subspace). It is still an open problem how to characterize the invariant subspaces of A^p.

One of the motivations for studying invariant subspaces of Bergman spaces is the potential application in operator theory. More specifically, the structure of the lattice of invariant subspaces for the Bergman space A^2 is related to the *invariant subspace problem* in operator theory, which asks whether or not every bounded linear operator T on an infinite dimensional Hilbert space H has an invariant subspace. Note that a closed subspace I of H is called an invariant subspace for T if T maps I back into itself. z-invariant subspaces of the Hilbert space A^2 are exactly the invariant subspaces for the operator $T = M_z$ of multiplcation by the coordinate function z. It is well known that an affirmative answer to the invariant subspace problem is equivalent to the validity of the following assertion: given any two invariant subspaces I_0 and I_1 of A^2, with the properties that $I_0 \subset I_1$ and

$\dim(I_1 \ominus I_0) = \infty$, there exists another invariant subspace I of A^2 that lies properly between I_0 and I_1.

More generally, invariant subspaces of the Bergman space A^2 are important in model theory for contractions (an important topic in modern operator theory). A linear operator T on a Hilbert space H is called a contraction if $\|T\| \leq 1$. It is possible to realize (up to unitary equivalence) "most contractions" as compressions (multiplication by z followed by an orthogonal projection back into the space) on spaces of the form $J \ominus I$, where $I \subset J$ are invariant subspaces of A^2.

Obviously, the understanding of the invariant subspace lattice of A^2 is extremely important, difficult, and appealing.

5. TOEPLITZ OPERATORS ON A^2

Since A^2 is a closed subspace of the Hilbert space $L^2(\Omega, dv)$, there exists an orthogonal projection

$$P : L^2(\Omega, dv) \to A^2(\Omega).$$

This is called the Bergman projection, and it is easy to show that it is an integral operator,

$$P(f)(z) = \int_\Omega K(z,w) f(w)\, dv(w),$$

where $K(z,w)$ is the Bergman kernel of Ω.

Given a bounded function φ on Ω, a bounded linear operator T_φ, called the Toeplitz operator with symbol φ, can be defined on A^2 as follows:

$$T_\varphi(f) = P(\varphi f), \quad f \in A^2.$$

The study of Toeplitz operators on A^2 has flourished in the last two decades, with the main focus being on the interplay between function theoretic properties of φ and operator theoretic properties of T_φ.

Here are several problems that are easy to state but remain open. The first conjecture states that there exists no finite-rank Toeplitz operators on A^2 except the zero operator. The second problem concerns products of Toeplitz operators, namely, do there exist nonzero Toeplitz operators T_{φ_k}, $1 \leq k \leq n$ with $n > 1$, such that

$$T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} = 0?$$

The third problem concerns spectral properties of T_φ. Except in very special situations, very little is known about the spectrum, essential spectrum, and even the norm of the Toeplitz operator T_φ.

6. THE BEREZIN TRANSFORM

Closely related to the notion of Toeplitz operators on the Bergman space is the notion of the Berezin transform, whose counterpart in the more classical theory of Hardy spaces is the Poisson transform.

Suppose Ω has the property that the Bergman kernel is never zero. Then for any $w \in \Omega$ we can define a unit vector k_w in A^2 as follows:

$$k_w(z) = K(z,w)/\sqrt{K(w,w)}, \quad z \in \Omega.$$

This is called the normalized reproducing kernel of A^2 at w. Now if T_φ is a Toeplitz operator on A^2, then it induces a function $B(\varphi)$ on Ω as follows:

$$B(\varphi)(w) = \langle T_\varphi k_w, k_w \rangle, \quad w \in \Omega.$$

This is called the Berezin transform of φ.

Obviously, certain information about the operator T_φ is encoded in the Berezin transform $B(\varphi)$. In fact, it can be shown that the mapping $\varphi \mapsto T_\varphi \mapsto B(\varphi)$ is one-to-one, so at least in theory, $B(\varphi)$ contains all the information about φ.

Natural questions about the Berezin transform include the following: What are the fixed points of B? In the case of the unit disk, they are the harmonic functions. What is the range of the Berezin transform? Very little is known here. What is the spectrum in general, and the set of eigenvalues in particular, of the Berezin transform B when considered as an (often) bounded operator on $L^p(\Omega, dv)$?

7. HANKEL OPERATORS

For a bounded function φ on Ω the Hankel operator with symbol φ, denoted by H_φ, is defined on A^2 as follows:

$$H_\varphi(f) = (I - P)(\varphi f), \quad f \in A^2,$$

where P is the Bergman projection and I is the identity operator. Obviously, H_φ is a bounded linear operator from A^2 into $L^2(\Omega, dv) \ominus A^2$. It can be checked that

$$H_\varphi^* H_\varphi = T_{|\varphi|^2} - T_{\overline{\varphi}} T_\varphi$$

as operators on A^2.

The study of Hankel operators on the Bergman space has been a sizeable industry in the last two decades. A lot of results have been achieved in this area. In particular, the size of H_φ is closely related to the mean oscillation of φ in the Bergman metric. This relationship is well understood in the case when the domain Ω is nice, for example, when the domain is bounded symmetric or when the domain is strongly pseudo-convex. The situation is less clear when the domain is not so nice and a lot of open problems remain in this area.

8. COMPOSITION OPERATORS

If $\varphi : \Omega \to \Omega$ is holomorphic, then it induces a linear operator

$$C_\varphi : H(\Omega) \to H(\Omega)$$

as follows: $C_\varphi(f) = f \circ \varphi$. This is called the composition operator induced by φ.

In the case when $\Omega = \mathbb{D}$ is the unit disk, it is classical that C_φ is a bounded linear operator on each A^p, and there has been a lot of research on such composition operators. However, for more general domains (even for the unit ball or the polydisk in $\mathbb{C}^n$, $n > 1$), the operator C_φ is often unbounded on A^p. Several fundamental problems concerning such composition operators are still open. For example, there is no satisfactory characterization of bounded and compact composition operators on the Bergman space of the unit ball or the polydisk in $\mathbb{C}^n$ when $n > 1$, although there are several approaches based on Carleson type measures.

The invariant subspace problem is also related to composition operators. More specifically, if the invariant subspaces for composition operators induced by fractional linear transformations on the unit disk are fully understood, then a solution to the invariant subspace problem can be solved one way or another.