Research Interests of Mark Steinberger

Currently, my main interest is in *De Rham's Problem*, a.k.a. the *Nonlinear Similarity Problem*.

De Rham's Problem was originally stated for matrices. In this form, it has a particularly clean, elementary statement, which we give in the following link.

- De Rham's problem for matrices[1]

De Rham's problem may also be stated in terms of representations of finite groups. (The matrix statement corresponds to the case of cyclic groups.) We first take a moment to discuss representations.

Write $\mathrm{GL}_n(\mathbf{R})$ for the group (under matrix multiplication) of $n \times n$ invertible matrices over the real numbers $\mathbf{R}$. An n-dimensional real representation, ρ, of a group G is a group homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbf{R}) \quad .$$

Thus, for $g \in G$, the matrix $\rho(g)$ defines an operator on Euclidean n-space $\mathbf{R}^n$. By the representation space of ρ, we mean $\mathbf{R}^n$, with G acting on it via ρ as a group of operators.

Two representations, ρ and θ, are linearly equivalent (linearly similar) if there is an invertible matrix A such that

$$A\rho(g)A^{-1} \; = \; \theta(g)$$

for all $g \in G$. Representation theory shows that two representations of a finite group are linearly equivalent if and only if for each $g \in G$, the matrices $\rho(g)$ and $\theta(g)$ have the same trace, so linear equivalence is easy to detect.

Let ρ and θ be real representations of dimension n. We say that ρ and θ are topologically similar if there is a homeomorphism (i.e., a continuous, 1-1 and onto function whose inverse is continuous)

$$f : \mathbf{R}^n \to \mathbf{R}^n$$

which commutes with the actions of G in the sense that

$$f(\rho(g)x) \; = \; \theta(g)f(x)$$

for all $g \in G$ and x in $\mathbf{R}^n$. We call such an f a G-homeomorphism from the representation space of ρ to the representation space of θ. Note that since f is invertible, this says that, regarding $\rho(g)$ and $\theta(g)$ as operators, we have

$$f\rho(g)f^{-1} \; = \; \theta(g)$$

for all $g \in G$. (Thus topological similarity means that ρ and θ are similar as homomorphisms from G to the group of homeomorphisms from $\mathbf{R}^n$ to itself.)

Suppose ρ and θ are topologically similar via a G-homeomorphism

$$f : \mathbf{R}^n \to \mathbf{R}^n.$$

[1] URI: http://math.albany.edu/˜mark/derham.html

If ρ and θ are not linearly similar, we say f is a nonlinear similarity.

De Rham's Problem: When does there exist a nonlinear similarity between a given pair of linearly inequivalent representations?

In 1935, de Rham conjectured this couldn't happen. His conjecture was false, and the first counterexamples were found by Cappell and Shaneson in 1979.

Hsiang and Pardon, and, independently, Madsen and Rothenberg, showed in 1980 that nonlinear similarity may only occur for groups whose order is properly divisible by 4.

My own contributions to this problem (including joint work) include the following:

- A pair of linearly inequivalent representations is nonlinearly similar if and only if their unit spheres are h-cobordant in the category of locally linear G-manifolds [1]. The classification of such h-cobordisms, also given in [1], has been applied in [1], as well as the papers below, to obtain applications regarding the nonlinear similarity problem.
- "Nonlinear similarity begins in dimension 6" [2], i.e., 6 is the minimal dimension in which nonlinear similarity may occur.
  - Some examples of six-dimensional nonlinear similarities[2].
- Nonlinear similarities of cyclic 2-groups are classified in [3].

The six-dimensional examples have an interesting property. By [1], (and by construction) their unit spheres are G-h-cobordant in the category of locally linear actions. But if these G-h-cobordisms had G-smoothings of any kind, then an analysis of the tangent representations and the exponential map would produce five dimensional nonlinear similarities. Since such similarities cannot exist, the study of non-smoothable G-h-cobordisms of [1] is essential to an understanding of the Nonlinear Similarity Problem.

The above results are based on theoretical results regarding locally linear actions of finite groups on manifolds, as well as on calculations in algebraic K-theory and surgery theory[3].

I have always enjoyed calculational mathematics. Earlier in my career, I did some calculations of homology operations for spectra with enriched ring structure [4].

References

[1] M. Steinberger, *The equivariant topological s-cobordism theorem*, Invent. Math., **91** (1988), 61-104.

[2] S.E. Cappell, J.L. Shaneson, M. Steinberger and J.E. West, *Nonlinear similarity begins in dimension six*, Amer. J. Math. **111** (1989), 717-752.

[3] S.E. Cappell, J.L. Shaneson, M. Steinberger, S. Weinberger and J.E. West, *The classification of nonlinear similarities over* $\mathbf{Z}_{2^r}$, Bull AMS. **22** (1990), 51-57.

[2] URI: http://math.albany.edu/~mark/simexs.html
[3] URI: http://math.albany.edu/~mark/surgery.htm

- Preprint version: dvi[4] postscript[5]

[4] R. Bruner, J.P. May, J. McClure, and M. Steinberger, *H_∞ Ring Spectra and their Applications*, Lecture Notes in Mathematics vol. 1176, Springer-Verlag, Berlin, 1986.

Mark Steinberger's home page[6]

[4]URI: http://math.albany.edu/~mark/cssww.dvi
[5]URI: http://math.albany.edu/~mark/cssww.ps
[6]URI: http://math.albany.edu/~mark/index.html