Department of Mathematics and Statistics

Ph.D. Preliminary Examination in Real Analysis

January 18, 1994

Do as many as time permits.

1. The following theorems are easily shown to imply one another:

   i) The Lebesgue Dominated Convergence Theorem.

   ii) Fatou's Lemma.

   iii) The Lebesgue Monotone Convergence Theorem.

   State the three above theorems, pick one of them, and prove it. (Of course, not using the other two or some other nearly equivalent theorem.)

2. Give an example of a sequence of non-negative continuous functions (f_n) defined on [0,1] such that

   (a) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \infty$.

   (b) $\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{1+f_n(x)}dx = 0$.

   Verify (a) and (b).

3. Prove that the characteristic function of the Cantor set is Riemann integrable.

4. Prove that $\lim_{n\to\infty} \int_0^1 x \sin nx \; dx = 0$.

5. Let m be Lebesgue measure on $X = [0,1]$, and let μ be a measure on the Lebesgue sets with $\mu(X) = 1$, and $\mu \sim m$ (i.e. μ and m have the same sets of measure zero). Prove there exists a measurable set A such that $\mu(A) = 1/2$.

6. Define a sequence of measures (μ_n) on the Lebesgue measurable subsets of [0,1] by $\mu_n(A) = \int_0^1 I_A(x)\ nx^{n-1}\ dx,\ n = 1, 2, 3, \ldots$.

(a) Verify that $\lim_{n\to\infty} \mu_n([a, b]) = 0$ if $0 < a < b < 1$.

(b) Suppose $0 < a_k < b_k < 1$ and $[a_k, b_k]$ are disjoint, $k = 1, 2, \ldots, r$. Verify that $\lim_{n\to\infty} \mu_n(\bigcup_{k=1}^{r} [a_k, b_k]) = 0$.

(c) Suppose $[a_k, b_k], k = 1, 2, 3, \ldots$, are all disjoint. Does $\lim_{n\to\infty} \mu_n(\bigcup_{k=1}^{\infty} [a_k, b_k])$ exist? Give reasoning for your answer.

7. Let λ be Lebesgue measure and μ be counting measure both regarded as Borel measures on $I = [0, 1]$. Let Δ be the diagonal in $I \times I$; $\Delta = \{(x, y)|x = y\}$.

(a) Show that Δ is measurable (with respect to the product measure on Borel subsets of $I \times I$.

(b) Let f be the characteristic function of Δ. Compute the integrals: $\int_I (\int_I f d\lambda) d\mu$, $\int_I (\int_I f d\mu) d\lambda)$ and $\int_{I\times I} f d\mu \times d\lambda$.

(c) Reconcile with Fubini's Theorem.

8. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$. Let (A_n) be a sequence of measurable sets with $\mu(A_n) = 1/2$, all n, and $\mu(A_n \cap A_m) = 1/4$, $n \neq m$. Let I_i be the characteristic function of a set A_i and let $f_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_i(x)$. Prove that

$$\lim_{n\to\infty} \int_X |f_n(x) - \frac{1}{2}| d\mu = 0 \ .$$

Hint: Show $\|f_n - 1/2\|_2^2 \to 0$ and use Hölder's Inequality.