Research Statement
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Overview

Most of my research is situated at the interface of algebraic combinatorics, representation theory, algebraic geometry, and algebraic topology. A unifying theme of this research is the emphasis on combinatorics and computation. Combinatorics could be described as the study of discrete configurations, and is, in particular, concerned with the enumeration of arrangements of objects according to specified rules. The increasing role of combinatorics in modern mathematics is due to the fact that combinatorial structures are particularly well suited for encoding complex mathematical objects, while combinatorial methods are well suited for related computations.

My work is part of an ongoing effort to perform concrete computations, to understand concrete spaces with combinatorial precision, and to build on the growing number of interactions between combinatorics and other areas of mathematics (surveyed in my paper [35]). These current trends in mathematics are due to the development of computer science, on the one hand (which stimulated research on discrete structures and related problems, and gave rise to experimental mathematics), and to developments within mathematics itself, on the other hand. Indeed, as A. Björner and R. Stanley point out in [BS99], "after an era where the fashion was to seek generality and abstraction, there is now much appreciation for concrete calculations, which are the *hard* problems".

Combinatorial structures are also very well suited for computer experiments, which usually give a better insight into mathematical phenomena. Computation based on combinatorial structures is a major part of my research, from the stage of computer experimentation (which I am actively pursuing in discovering and testing new methods of computation), to the stage of deriving/proving results, including explicit formulas, theorems, algorithms, and combinatorial models.

My research can be divided into the following areas.

(1) The representation theory of Lie groups and Lie algebras, particularly combinatorial representation theory.
(2) Modern Schubert calculus.
(3) The combinatorics of certain formal group laws related to algebraic topology.

In the past, I also did research in pattern recognition, mainly on mathematical aspects related to clustering and learning, plus related applications. This work is contained in [23, 24, 25, 34], as well as other publications.

I will start with an overview of the first three areas above, and will then give more details about my results and projects. Numbered citations are publications I authored or co-authored; the numbering coincides with that in my CV. The work I have done since August 2004 has been partially supported by two 3-year grants from the National Science Foundation. I also supported one graduate student during one summer from my first grant, and one student for three years from my second grant.

Representation theory is a fundamental tool for studying group symmetry – geometric, analytic, or algebraic – by means of linear algebra. It studies the way in which a given group may act on vector spaces; in other words, it is concerned with representing groups as groups of matrices. Since the first fundamental results in this area about a century ago, important advances have been made, through the study of representations of more and more general groups, through the discovery of geometric constructions of representations (which led to a new area: geometric representation theory), and through a better understanding of the subtle combinatorics involved, which led to some very explicit constructions and computations. Today, representation theory plays an important role in many recent developments of mathematics and theoretical physics. The physical connections are mainly related to the areas I am interested in, namely the representation theory of Lie groups/Lie algebras and quantum groups (which are deformations of universal enveloping algebras of Lie algebras depending on a "quantum parameter" q).

Classical Schubert calculus is concerned with certain enumerative problems in geometry, such as counting the lines or planes satisfying a number of generic intersection conditions. This is equivalent to performing a calculation in some algebraic structure (i.e., a cohomology ring) associated to the space of potential solutions, such as the Grassmannian of k-dimensional subspaces in the n-dimensional complex space $\mathbb{C}^n$. Such classical problems have been generalized in several directions: (1) replacing the Grassmannian by more general varieties, such as a flag variety, a generalized flag variety G/B for a semisimple Lie group G, or a flag variety for a Kac-Moody group; (2) replacing ordinary cohomology by more general cohomology theories, such as K-theory, quantum cohomology, or T-equivariant cohomology/K-theory. In many of these cases, very little is known about the corresponding intersection theory. My work addresses some of the problems in this area.

Flag varieties are classical spaces, yet their remarkable combinatorial complexity is far from being understood, and there is considerable interest recently in the subtle interplay between various areas connected to them. Beside enumerative and algebraic geometry, that were mentioned above, other such areas are: representation theory, algebraic topology, symplectic geometry, commutative algebra, and algebraic combinatorics. In particular, flag varieties are closely related to the representation theory of Lie groups through constructions of group representations based on these varieties; this is another reason why I am particularly interested in them. Flag varieties also provide a useful testbed for the development of combinatorial models relevant to computations in various cohomology theories of a larger class of projective varieties. The idea is to encode the geometric information into some combinatorial structure (usually a graph or a partially ordered set, such as the Bruhat order on a Weyl group, in the case of flag varieties), and then to perform computations based on this structure (cf. [GKM98]). One of my long-term goals is to extend some of the results related to flag varieties to other interesting projective varieties with the action of a torus, which can also be modeled combinatorially based on graphs.

Another part of my research, related to formal group laws, is underlied by the same idea of performing concrete computations, this time with applications to problems in algebraic topology. Formal group laws represent a powerful machinery used by topologists to classify topological spaces. Studying the combinatorics of the coefficients of formal group laws brings new insights to this area. This research is only in an incipient stage. Interesting connections with enumerative combinatorics, as well as combinatorial Hopf algebras – to name just a few – have already emerged, and have been explored in some of my papers.

1. Combinatorial representation theory

I am working in several areas of combinatorial representation theory, as described below.

1.1. **Crystal bases.** These bases arose in the early 90s in the representation theory of quantum groups (the work of Kashiwara [Kas90, Kas91] and Lusztig [Lus90, Lus93]), as well as exactly solvable lattice models in statistical physics. Crystal bases can be viewed, intuitively, as bases for representations of a quantum group "at zero temperature". In this limit of the quantum parameter $q \to 0$, such a basis has a simple combinatorial structure, called a crystal, yet which captures the essential information of the representation. The usual representation theory of Lie groups is obtained in the limit $q \to 1$. It is an amazing result of Lusztig [Lus88] that the main questions of representation theory can be studied in either limit. The crystal limit lends itself to a beautiful combinatorial analysis.

A crystal is simply a directed graph with colored edges. The information encoded by a crystal is sufficient for solving classical problems in representation theory, such as decomposing tensor products of representations and deriving branching rules (for restricting representations to lower rank algebras).

In order to work with crystals, we need effective realizations of them. In the joint work with Postnikov [3, 5], we introduced a combinatorial model for crystals of semisimple Lie algebras and, more generally, of symmetrizable Kac-Moody algebras. We called this model the *alcove model*, because, in the case of semisimple Lie algebras, it is in terms of the corresponding affine Weyl group and the related alcove picture. On another hand, it is a discrete counterpart of the celebrated Littelmann path model [Lit94, Lit95, Lit97]. The alcove model has advantages over the Littelmann model and other models for crystals due to its generality, simplicity, combinatorial nature, and diverse applications (see below). In particular, the related computations are very explicit and straightforward, since they only involve enumerating certain saturated chains in the Bruhat order on the corresponding Weyl group. A reviewer of one of my NSF proposals had the following comment about the alcove model: "This beautiful model has the potential of successfully competing with the celebrated (but notoriously complicated) Littelmann's path model in the representation theory of semisimple Lie groups".

In the joint paper [32] with my student W. Adamczak, we construct bijections between the alcove model in the classical Lie types and the corresponding *Kashiwara-Nakashima tableaux*, which are certain filling of Young diagrams [KN94]; these bijections preserve the corresponding crystal structures. (By Lie types, I mean the Cartan-Killing classification of simple Lie algebras into classes labeled A to G, where the classical types A, B, C, D refer to the special linear, orthogonal, and symplectic Lie algebras; in addition, infinite types refer to the similar classification of affine Kac-Moody algebras.) In [6], I used the alcove model to give a type-independent combinatorial realization of *Lusztig's involution* [Lus93] on a crystal for a semisimple Lie algebra, which exhibits it as a self-dual partially ordered set; this is the first direct generalization of a well-known involution on *semistandard Young tableaux* (i.e., Kashiwara-Nakashima tableaux of type A), called *Schützenberger's involution* [Ful97]. In [4], I present a combinatorial description of the *commutor* in the category of crystals due to Henriques and Kamnitzer [HK06] (a commutor is an isomorphism between the crystals $X \otimes Y$ and $Y \otimes X$).

Several researchers have also been using the alcove model. Recent applications involve areas of considerable interest at present, namely Macdonald polynomials (cf. the Ram-Yip formula in [RY08], and Section 1.2) and Kirillov-Reshetikhin crystals for exceptional types [JS09]. I have implemented the alcove model in `Maple`, and the package is available on my webpage [36]. A more recent implementation is also available, as part of the algebraic combinatorics component `Sage-Combinat` [SC09] of the rapidly growing open-source system `Sage` [S^{+}09].

I have two projects related to crystals. The first one is related to finding a more explicit model that corresponds naturally to the version of the alcove model for affine algebras in [3]. With my student, A. Lubovsky, we started to investigate this problem in the case of the basic crystal $B(\Lambda_m)$ for $\widehat{\mathfrak{sl}}_n$. We intend to construct a bijection preserving the corresponding crystal structures between the alcove model and the more explicit Misra-Miwa model [MM90]; the latter is in terms of n-*restricted partitions*, i.e., partitions $\lambda = (\lambda_0 \geq \ldots \geq \lambda_{l-1} > \lambda_l = 0)$ with $\lambda_i - \lambda_{i+1} < n$. This map should extend to arbitrary highest weight crystals, for which we expect to retrieve the *abacus model* and the *cylindric plane partitions* in [Tin08]. It should then extend to other affine types (for which no analogues of the above partition models were yet found). This project would have various applications, cf. [KLMW05] and [AKT08], for instance to basis constructions for affine algebra representations.

My second project involving crystals is related to a bijection between the alcove model and the model based on *Mirković-Vilonen (MV) polytopes* [And03, Kam05, Kam07]. The latter are the images under the moment map of *MV-cycles* – a celebrated geometric basis for representations of reductive groups [MV07]. A direct connection between the alcove model and MV-polytopes is bound to be intricate because MV-polytopes are of a different combinatorial nature (polytopes versus paths), and they encode different information (namely, the Lusztig and the string parametrizations of the canonical basis, plus the so-called BZ-datum [BZ01], cf. the work of Kamnitzer [Kam05, Kam07]). This project would allow us to translate parameters encoded by one model into the other model.

1.2. **Macdonald and Hall-Littlewood polynomials.** Macdonald [Mac92, Mac01] defined a remarkable family of orthogonal symmetric polynomials $P_\lambda(x; q, t)$ associated to a finite root system, which bear his name. They specialize to the *Hall-Littlewood polynomials* (or spherical functions for a Chevalley group over a p-adic field) [Mac71] upon setting $q = 0$; they further specialize to the corresponding irreducible characters upon setting $t = 0$ as well. The importance of Macdonald polynomials is due to their deep connections with other areas of mathematics, such as: statistical physics, double affine Hecke algebras, and Hilbert schemes.

A combinatorial formula for the Hall-Littlewood polynomials $P_\lambda(x; t)$ of arbitrary type was given in terms of the alcove model by Schwer [Sch06], cf. also [Ram06]. On another hand, in type A, an apparently unrelated formula for $P_\lambda(x; t)$ follows from the Haglund-Haiman-Loehr (HHL) formula for Macdonald polynomials [HHL05], being based on a certain set of fillings of the Young diagram λ.

In [28] I show that the two formulas above are closely related. Namely, in the case of regular weights λ (i.e., partitions with no parts repeated), we can group the terms in Schwer's formula into classes, such that the sum in each class is a term in the formula based on fillings. This compression phenomenon explains the way in which a certain intricate statistic on fillings in [HHL05] (called "inv") follows naturally from more general concepts. I study the compression phenomenon for non-regular weights in the joint paper [31] with my student A. Lubovsky, for Macdonald polynomials in [2], and for Hall-Littlewood polynomials of type B and C in [27]. In the Macdonald case, I compress the recent Ram-Yip formula [RY08] and derive the HHL formula [HHL05], in terms of fillings. In types B and C, the HHL-type formulas I derive are new; the compression is quite large, e.g. by a factor of 45 for $\lambda = (3, 2, 1, 0)$ in type C_4.

My projects in this area are related to: (1) studying other instances of the compression phenomenon and deriving other compressed formulas, in terms of fillings, from the Ram-Yip formula; (2) using the compressed formulas in order to derive combinatorial formulas for the energy function in statistical physics (with C. Lecouvey and A. Schilling); (3) deriving a combinatorial formula for expanding the product of two Hall-Littlewood polynomials (with C. Lecouvey). The latter formula would generalize the well-known *Littlewood-Richardson rule* giving the tensor product decomposition of irreducible $\mathfrak{sl}_n$-modules.

1.3. **q-analogues of weight multiplicities for Lie superalgebras.** In [1], we define and study a generalization of *Lusztig's q-analogue of weight multiplicities* [Lus83] to the general linear Lie superalgebra $\mathfrak{gl}(n, m)$ and the orthosymplectic superalgebra $\mathfrak{spo}(2n, M)$. We prove the positivity property of this q-analogue, and exhibit a positive combinatorial formula in the case of $\mathfrak{gl}(n, m)$ by generalizing the *charge* statistic in [LS79].

1.4. **Combinatorial basis constructions for Lie algebra representations.** Crystals only partially encode the structure of a representation (they encode what happens "at zero temperature", as explained in Section 1.1). In order to completely recover the action of a Lie algebra on a basis, we need a more complex structure, known as a representation diagram; this is a graph on the same vertices as the crystal graph but containing more (colored) edges, plus complex numbers as edge labels. The theory of representation diagrams was developed in [Don03, Don97].

For $\mathfrak{sl}_n$, the celebrated *Gelfand-Tsetlin basis* [GC50] is the only known basis for which the Lie algebra action is given by explicit formulas. All the known proofs of the Gelfand-Tsetlin construction use sophisticated algebraic tools. Using the setup of representation diagrams, in the joint paper with Hersh [29], we rederived the construction in a simple way, together with a certain minimality property of the corresponding graph.

In collaboration with C. Lecouvey, I will work on representation diagrams in type C, based on the similar results in type A in [29], and on the combinatorics of type C Kashiwara-Nakashima tableaux (see Section 1.1) studied by Lecouvey [Lec02].

2. Modern Schubert calculus

The object of study in modern Schubert calculus is the generalized flag variety G/B, where G is a connected, simply connected, semisimple complex Lie group, T a maximal torus, $B \supseteq T$ a Borel subgroup, and B^- the opposite Borel subgroup. In type A, SL_n/B is the variety Fl_n of complete flags $(0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$. Let Φ be the corresponding root system, and Φ^+ the corresponding set of positive roots. The Bruhat order on the Weyl group W is the transitive closure of $w < ws_\alpha$, where $\alpha \in \Phi^+$ and the length increases, that is, $\ell(w) < \ell(ws_\alpha)$. For each w in W, the subset $X_w^\circ := B^- wB/B$ is the corresponding Schubert cell, and its closure X_w is the corresponding Schubert variety, of complex codimension $\ell(w)$. We denote by σ_w the cohomology class of X_w, which is in $H^{2l(w)}(G/B)$. The collection of all these classes forms a basis of $H^*(G/B)$ (cf. [BGG73]).

The central problem in this area, on which I am working, is understanding the multiplication of Schubert classes, that is, studying the intersection theory on G/B. More precisely, we have

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w \,, \tag{1}$$

where the structure constants c_{uv}^w are nonnegative integers, since they count points in a suitable triple intersection of Schubert varieties. Finding a combinatorial interpretation for c_{uv}^w (and, in particular, a proof of their nonnegativity which bypasses geometry) is known as the Littlewood-Richardson problem for the cohomology of G/B, since it is a generalization of the classical Littlewood-Richardson rule for multiplying *Schur polynomials* (the latter represent Schubert classes in the cohomology of a Grassmannian). The importance of this problem stems from the geometric significance of the mentioned coefficients (also known as Littlewood-Richardson coefficients), and from the fact that a combinatorial interpretation for them would facilitate a deeper study of their properties (such as their symmetries, vanishing etc.). The Littlewood-Richardson problem proved to be a very hard problem, resisting many attempts to be solved, and is mentioned as Problem 11 in Stanley's survey of open problems in algebraic combinatorics [Sta00]. This problem generalizes to K-theory, quantum cohomology, and T-equivariant cohomology/K-theory. For instance, the Schubert classes in K-theory are the classes of structure sheaves of Schubert varieties, and their multiplication generalizes the multiplication of Schubert classes in cohomology; indeed, the Littlewood-Richardson coefficients in cohomology are a subset of the Littlewood-Richardson coefficients in K-theory.

In order to address the Littlewood-Richardson problem, it is useful to have polynomial representatives for Schubert classes. In type A these are the *Schubert polynomials* (for cohomology) and *Grothendieck polynomials* (for K-theory) due to Lascoux and Schützenberger [Las90, LS82a, LS82b].

These polynomials are indexed by permutations in the symmetric group S_n. They have a rich combinatorics, which generalizes the one of semistandard Young tableaux [Ful97], corresponding to Schur polynomials.

2.1. **The cohomology of the classical flag variety.** In [11], I investigate some of the connections not yet understood between several combinatorial structures for the construction of Schubert polynomials; I also present simplifications in some of the existing approaches to this area based on new combinatorial concepts. In particular, I introduce a crystal-like structure (cf. Section 1.1) on one particular family of combinatorial objects in this area.

In [16, 17], I extend the work of Fomin and Greene on noncommutative Schur functions [FG98] (which is a useful tool for proving certain positivity results for symmetric functions) by defining noncommutative analogues of Schubert polynomials. A new combinatorial procedure for them implies noncommutative analogues of certain formulas in the theory of Schubert polynomials (namely, the Pieri formula and the Cauchy identity [Mac91, Man01]). As an application, I prove the positivity conjecture of Fomin and Kirillov concerning the expansion of an arbitrary Grothendieck polynomial in the basis of Schubert polynomials.

In [12], we generalize skew Schur (symmetric) functions by defining skew Schubert polynomials. We show that our skew Schubert polynomials expand in the basis of Schubert polynomials with nonnegative integer coefficients that are precisely the type A structure constants c_{uv}^{w} in (1). This result leads to a new approach to the Littlewood-Richardson problem for the classical flag variety. The results in this paper were extended to quantum cohomology by Postnikov in [Pos05].

The study of the action of the *Steenrod algebra* on the mod p cohomology of spaces has many applications to the topological structure of those spaces. In [18], I present several combinatorial formulas for the action of Steenrod operations on the cohomology of Grassmannians. These formulas are based on the combinatorics of symmetric functions, as well as on certain integral lifts of Steenrod operations, which lie in a Hopf algebra of differential operators [Woo98]. This paper was cited in [Woo98], which is the most comprehensive survey on the Steenrod algebra.

In [26] I propose a new approach to the Littlewood-Richardson rule, based on generalizing Fomin's *growth diagrams* from chains in Young's lattice of partitions [Sta99][Appendix 1] to chains in the Bruhat order on S_n. The advantage is the potential for generalizing Thomas and Yong's S_3-symmetric rule [TY08] for the Grassmannian structure constants, which is based on Fomin's growth diagrams. More precisely, I give a combinatorial interpretation for certain type A Littlewood-Richardson coefficients that include the Grassmannian ones. I intend to extend these results to a larger class of Littlewood-Richardson coefficients.

Vakil [Vak06] and Coskun [Cos09] were the first to give geometric Littlewood-Richardson rules for the multiplication of Schubert classes in the cohomology of the Grassmannian. These rules, based on degeneration techniques, were combinatorialized [Knu07, Liu08, Vak06], and have bijections to the classical Littlewood-Richardson rule [Ful97]. However, the iterative degeneration steps are not understood in terms of the usual tableau combinatorics. I am currently investigating a way of to solve this problem and to generalize it, based on the growth diagram approach in [26].

2.2. **The K-theory of the classical flag variety.** In [9], we give several new formulas for Grothendieck polynomials. In particular, we give substitution formulas and a combinatorial construction of Grothendieck polynomials in terms of chains in the Bruhat order on the symmetric group. One of our substitution formulas was used in [KY04] to attack the Littlewood-Richardson problem for Grothendieck polynomials.

In [7], we derive explicit formulas, with no cancellations, for expanding in the basis of Grothendieck polynomials the product of two such polynomials, one of which is indexed by an arbitrary permutation, and the other by a cycle of the form $(k-p+1, k-p+2, ..., k+1)$ or $(k+p, k+p-1, ..., k)$. These are Pieri-type formulas, generalizing the Monk-type formula in [13]. They express the product in the Grothendieck ring of the flag variety between an arbitrary Schubert class and certain special Schubert classes, pulled back from Grassmannian projections. Our formulas are in terms of certain labeled chains in the Bruhat order on the symmetric group.

The Pieri-type formulas in [7] also generalize the corresponding formulas in the K-theory of Grassmannians that were previously obtained in [14]. On the other hand, the latter formulas were used in a crucial way in Buch's work [Buc02b, Buc02a]. For instance, in [Buc02b], they are used in the proof of a combinatorial formula for multiplying any two Schubert classes in the K-theory of Grassmannians. This formula is a beautiful generalization of the classical Littlewood-Richardson rule. Buch also gives other applications of formulas in [14].

2.3. **Schubert calculus for generalized flag varieties.** Let us now turn to the K-theory of generalized flag varieties, where G is a complex semisimple Lie group. The Schubert classes, which form a basis of the K-theory of G/B, are the classes of structure sheaves $\mathcal{O}_w = \mathcal{O}_{X_w}$ of Schubert varieties X_w, for w in the Weyl group W.

In [5] we gave a very general Chevalley-type multiplication formula in the T-equivariant K-theory $K_T(G/B)$, where T is the corresponding torus. This is a formula for multiplying the class of $\mathcal{O}_w$ with the class of the line bundle $\mathcal{L}_\lambda$ associated to an arbitrary weight λ. Our formula is based on the alcove model (cf. Section 1.1), and is expressed in terms of chains in the Bruhat order on W. It is the natural generalization of my previous type A formula in [13]. Based on the mentioned K-Chevalley formula, in [8] we give a model for $K_T(G/B)$ in terms of a certain braided Hopf algebra called the *Nichols-Woronowicz algebra*. This result generalizes my earlier work [10] and belongs to the research initiated by Fomin and Kirillov [FK99] on the realization of the cohomology and K-theory of G/B as commutative subalgebras of certain noncommutative algebras, cf. [Baz06, KM04, KM05a, KM05b].

With M. Shimozono, we are working on a generalization of the K-Chevalley formula in [5] to the infinite case, namely to the equivariant K-theory of a flag manifold corresponding to a Kac-Moody group [Kas89]. Based on the extension of the alcove model to the infinite case in [3], I conjectured such a formula, which looks completely similar to its finite counterpart, being based on chains in the Bruhat graph on the corresponding Weyl group.

2.4. **Quantum cohomology and quantum K-theory.** Let us now consider the quantum cohomology $QH^*(Fl_n)$ of the type A flag variety. This is a certain quotient of

$$\mathbb{Z}[q, x] := \mathbb{Z}[q_1, \ldots, q_{n-1}] \otimes \mathbb{Z}[x_1, \ldots, x_n] \,.$$

Fomin, Gelfand, and Postnikov [FGP97] defined polynomial representatives in $\mathbb{Z}[q, x]$, called *quantum Schubert polynomials*, for the quantum Schubert classes in $QH^*(Fl_n)$, for $w \in S_n$. They also gave a quantum Chevalley formula for multiplying an arbitrary quantum Schubert polynomial with one indexed by an adjacent transposition (simple reflection). The formula is in terms of the *quantum Bruhat graph*, which is constructed by adding to the graph of the Bruhat order certain "quantum" edges.

In [30], we extend the ideas in [FGP97] to the quantum K-theory $QK(Fl_n)$ (cf. [Lee04, GL03]). We define *quantum Grothendieck polynomials* by extending the quantization map approach to the construction of quantum Schubert polynomials in [FGP97]. Our quantization map is substantially more complex than its quantum cohomology counterpart; its construction is based on computations with the Toda lattice due to Kirillov and Maeno [KM]. Using results in [7], we proved a "quantum K-Chevalley formula", which is the natural generalization of the corresponding formulas (mentioned above) in K-theory [5] and quantum cohomology [FGP97]. We conjecture that our quantum

Grothendieck polynomials represent Schubert classes in $QK(Fl_n)$. Currently, Buch and Mihalcea are working on this conjecture and have some preliminary results.

2.5. **MV-cycles, GKM theory, and affine Schubert calculus.** I briefly mention a long-term, multi-faceted joint project with T. Braden, J. Kamnitzer, T. Lam, and J. Tymoczko. In Section 1.1, I mentioned that MV-cycles [MV07] are a celebrated geometric basis for representations of a reductive group G. They lie in the intersection cohomology $IH^*(Gr^\lambda)$ of a certain subvariety Gr^λ of the affine Grassmannian $Gr = Gr_G$ of G, where $IH^*(Gr^\lambda)$ is isomorphic to the irreducible representation of G of highest weight λ, as shown by Lusztig [Lus83]. Our goal is to understand the combinatorics of MV-cycles and related structures. Ultimately, we would like to describe explicitly the action of the Lie group on this important geometric basis, as this action is currently only defined in geometric terms, not by an explicit formula.

We propose to address this goal by using new tools, from the rapidly growing areas of GKM theory and affine Schubert calculus. GKM (Goresky-Kottwitz-MacPherson) theory is a combinatorial algorithm for computing T-equivariant cohomology for a complex algebraic variety with a suitable action of a torus T [GKM98]. Affine Schubert calculus mainly refers here to the study of Schubert classes in the (co)homology of Gr. This field has experienced an impressive growth in recent years, particularly through the work of Lam and his collaborators [Lam08, LLMS06, LSS07, LSS09]. The alcove model (cf. Section 1.1), which is defined, like MV-cycles, in an affine setup (the affine Weyl group versus the affine Grassmannian), is also relevant to this work.

3. Combinatorics of formal group laws

My research in formal group theory is underlied by the same idea of performing concrete computations based on combinatorial models.

A (one-dimensional, commutative) formal group law over a commutative ring R is a formal power series $F(x,y)$ in $R[[x,y]]$ with the following properties: (1) $F(x,0) = F(0,x) = x$; (2) $F(x,y) = F(y,x)$; (3) $F(x,F(y,z)) = F(F(x,y),z)$. The simplest example of a formal group law (after the trivial one $x+y$) is the multiplicative one $x+y-xy$. By considering the pair $(R, F(x,y))$, one can define the category of such objects; more precisely, the morphisms are ring homomorphisms whose extensions to the corresponding rings of formal power series map one formal group law to another. The universal ring is called the Lazard ring. A celebrated theorem of Lazard states that this ring is isomorphic to $\mathbb{Z}[x_1, x_2, \ldots]$ [Haz78, Rav86].

The structures defined above have a close connection to algebraic topology. Indeed, any generalized cohomology theory $E^*(\cdot)$ has a formal group law naturally associated with it. For instance, for ordinary cohomology $H^*(\cdot)$ we have the trivial formal group law, for K-theory we have the multiplicative one, for complex cobordism $MU^*(\cdot)$ we have the universal formal group law.

In [21, 22, 33], we studied the combinatorics related to the coefficients of the (iterated) universal formal group law. This work was based on the connection between formal group theory and the combinatorics related to incidence Hopf algebras [DRS72, Sch94] via umbral calculus [RT93, RT94].

In [19], I give a shorter proof of Lazard's theorem than the classical one, based on a new approach, which involves the combinatorics of symmetric functions. I also defined and studied new polynomial generators for the Lazard ring.

Other combinatorial results related to formal group laws are contained in my paper [20]. This is devoted to the construction and study of a generalization of the *necklace algebra* defined by Metropolis and Rota [MR83]. The latter was introduced in order to simplify the construction of the universal ring of *Witt vectors* (associated with a commutative ring), which attracted the interest of many mathematicians. My generalized necklace algebra corresponds to an arbitrary formal group law, whereas the one of Metropolis and Rota corresponds to the multiplicative formal group law $x+y-xy$. A q-deformation of the classical necklace algebra is defined by considering the formal group law $x+y-qxy$. The constructions in [20] were rephrased in the language of profinite groups and categories in a series of papers by Y.-T. Oh [Oh08a, Oh08b, Oh07c, Oh07a, Oh07b, Oh06, Oh05].

The substitutional inverse of formal power series (also known as *Lagrange inversion*) plays an important role in the construction of formal group laws. In [15], I study the involution on the algebra of symmetric function defined by Macdonald [Mac95], which is based on Lagrange inversion. More precisely, I prove a positivity result related to the Schur function expansion of the images of skew Schur functions under the mentioned involution. A q-analogue of this result is also presented, and it turns out to be a special case of a conjecture of Bergeron et al. concerning their operator ∇ [BGHT99]. The latter was defined in the theory of Macdonald polynomials. The results in my paper were generalized in [LW08].

A project of mine in the general area above is to use the combinatorics of the universal formal group law mentioned above in order to study an extension of Schubert calculus (cf. Section 2) to complex cobordism $MU^*(\cdot)$. This was first considered in a 1992 paper of Bressler and Evens [BE92], but then there was no work in this direction until some recent results in [HK09, CPZ09].

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