Review of Curve and Surface Integrals

for Calculus III (Math 214)

Definitions

Curve integrals

C: $\mathbf{r}(t),\ a \le t \le b$

(a) scalar function f:

$$\int_C f\, ds \ = \ \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt$$

(b) vector function F:

$$\int_C F \cdot d\mathbf{r} \ = \ \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Note: $d\mathbf{r} \ = \ \mathbf{r}'(t)\, dt \ = \ \mathbf{T}\, ds; \quad ds \ = \ \|d\mathbf{r}\| \ = \ \|\mathbf{r}'(t)\|\, dt$

Surface integrals

S: $\mathbf{W}(u,v),\ (u,v) \in E \qquad$ (E a planar region)

(a) scalar function f:

$$\iint_S f\, d\sigma \ = \ \iint_E f(\mathbf{W}(u,v))\, \|\mathbf{W}_u \times \mathbf{W}_v\|\ dudv$$

(b) vector function F:

$$\iint_S F \cdot d\mathbf{W} \ = \ \iint_E F(\mathbf{W}(u,v)) \cdot \ (\mathbf{W}_u \times \mathbf{W}_v)\ dudv$$

Note: $d\mathbf{W} \ = \ (\mathbf{W}_u \times \mathbf{W}_v)\, dudv \ = \ \mathbf{N} d\sigma; \quad d\sigma \ = \ \|d\mathbf{W}\| \ = \ \|\mathbf{W}_u \times \mathbf{W}_v\|\, dudv$
where

$$W_u \ = \ \frac{\partial W}{\partial u}\ , \quad W_v \ = \ \frac{\partial W}{\partial v}$$

Special case 1: graph of a function of two variables

$$S:\ z \ = \ f(x,y), \quad (u,v) \ = \ (x,y), \quad \mathbf{W}(x,y) \ = \ (x,y,f(x,y)), \quad d\mathbf{W} \ = \ (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) dxdy$$

Special case 2: a sphere parameterized with spherical coordinates

$$\mathbf{W}(u,v) \ = \ a(\cos u\, \sin v, \sin u\, \sin v, \cos v) \quad d\mathbf{W} \ = \ -a^2 \sin v(\cos u\, \sin v, \sin u\, \sin v, \cos v)\, du\, dv$$

For both curves and surfaces a parameterization determines an orientation. The orientation of a surface in $\mathbf{R}^3$ is determined by the two-fold choice of a unit normal to that surface, and the boundary of an oriented surface is given the orientation that is related to the chosen unit normal in a right-handed coordinate system by the "right-hand rule".

The Fundamental Theorem of Calculus

Theorems of the form $\int_{\partial G} \omega = \int_G d\omega$

dim G	$\boldsymbol{\omega}$	$\boldsymbol{d\omega}$	**left side**	**right side**	**Remarks**
1	f	f'	$f(b) - f(a)$	$\int_I f'(t)\,dt$	**Fund. Thm. of Calculus** I interval in $\mathbf{R}$ from a to b
1	f	∇f	$f(B) - f(A)$	$\int_C \mathrm{grad} f \cdot d\mathbf{r}$	C path in $\mathbf{R}^n$ from A to B
2	$F \cdot d\mathbf{r}$	$(\nabla \wedge F)\,dA$	$\int_{\partial R} F \cdot d\mathbf{r}$	$\iint_R \mathrm{curl} F\,dA$	**Green's Thm.** R region in $\mathbf{R}^2$ (∂R anti-clockwise in right-hand coord. system)
2	$F \cdot d\mathbf{r}$	$(\nabla \times F) \cdot d\mathbf{W}$	$\int_{\partial S} F \cdot d\mathbf{r}$	$\iint_S \mathrm{curl} F \cdot d\mathbf{W}$	**Classical Stokes' Thm.** S surface in $\mathbf{R}^3$ (with r/l hand rule)
3	$F \cdot d\mathbf{W}$	$(\nabla \cdot F)\,dV$	$\iint_{\partial D} F \cdot d\mathbf{W}$	$\iiint_D \mathrm{div} F\,dV$	**The Divergence Thm.** D domain in $\mathbf{R}^3$ (∂D with outer normal)