Modern Computing for Mathematicians (Math 587)

A generalization of the Syracuse iterator

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Recall that the Syracuse function s is defined for integers n by

$$s(n) \;=\; \begin{cases} 1 & \text{if } \; n \le 1 \\ 3n+1 & \text{if } \; n > 1 \; \text{ is odd} \\ n/2 & \text{if } \; n > 1 \; \text{ is even} \end{cases}$$

Many questions about the iterative behavior of s – in particular, the question of whether the value of some iterate from any starting point is 1 – are unaffected if s is replaced with the function s_1 defined as follows:

$$s_1(n) \;=\; \begin{cases} 1 & \text{if } \; n \le 1 \\ 3n+1 & \text{if } \; n > 1 \; \text{ is odd} \\ n/2^k & \text{if } \; n \;=\; 2^k m \text{ where } m \text{ is odd and } k \ge 1 \end{cases}$$

or, as one might more informally write, for $n \ge 1$,

$$s_1(n) \;=\; 3n+1 \text{ made coprime to } 2 \quad .$$

Generalizing this, for given pairwise coprime integers $a, b, m > 0$ with $m \ge 2$, one defines for $n \ge 1$

$$f_{<a,b,m>}(n) \;=\; an+b \text{ made coprime to } m \quad .$$

Here, for an integer x, the phrase "x made coprime to m" means that for any common prime divisor p of x and m the highest power of p dividing x is removed as a factor. Note that the meaning of $f_{<a,b,m>}$ is not changed when m is replaced by the product of the distinct primes dividing m; that is, without loss of generality one may restrict to the case where m is square-free.

Example: $s_1 \;=\; f_{<3,1,2>}$.

Exercises:

1. Write code for gp to investigate the iterates of $f_{<a,b,m>}$ from a given integer. In particular, the code should be able to determine whether from a given starting integer n a cycle is formed within the first N iterates.
2. Determine what cycles, if any, are formed and whether there seems to be a pattern of unbounded growth in the iterates for $n, N \le 10000$ in the following cases:
   (a) $f_{<3,2,5>}$.
   (b) $f_{<5,1,3>}$.
   (c) $f_{<17,1,30030>}$.