Math 520B
Written Assignment No. 5

due Friday, December 9, 2005

Directions. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises is **not** permitted. You may not seek help from others.

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1. Let f be the $\mathbf{Z}$-linear map from $\mathbf{Z}^4$ to $\mathbf{Z}^3$ defined by $f(x) = Mx$ for all $x \in \mathbf{Z}^4$ where M is the 3×4 matrix
$$M = \begin{pmatrix} 24 & 30 & -36 & -42 \\ 14 & 24 & -34 & -44 \\ 34 & 48 & -62 & -76 \end{pmatrix} .$$
Represent the cokernel of f as a direct sum of cyclic groups.

2. Find one matrix in each of the similarity classes of matrices over the field $\mathbf{C}$ (of complex numbers) that share the characteristic polynomial $t^3 - t^2 - t + 1$.

3. Let A be the 5×5 matrix in the field $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} .$$
Find a direct sum of companion matrices that is similar to A.

4. Let M be the 6×6 rational matrix
$$M = \begin{pmatrix} 102 & -52 & 71 & -21 & 16 & -16 \\ 324 & -157 & 228 & -69 & 78 & -57 \\ 357 & -182 & 244 & -66 & 49 & -60 \\ 532 & -286 & 356 & -88 & 19 & -82 \\ 36 & -26 & 20 & 3 & -24 & -6 \\ 532 & -286 & 356 & -90 & 19 & -80 \end{pmatrix} .$$
Find the sequence of successively divisible invariant factors as well as the minimal and characteristic polynomials of M.

5. Let V and W be n-dimensional vector spaces over a field F, and let $b : V \times W \to F$ be F-bilinear. The bilinear form b determines a linear map $\phi_b \in \mathrm{Hom}_F(V, \check{W})$, with $\check{W} = \mathrm{Hom}_F(W, F)$, that is defined by
$$\phi_b(v) = [w \longmapsto b(v, w)] .$$
   (a) Show that ϕ_b is an isomorphism if and only if for given bases $v_1, \ldots, v_n$ of V and $w_1, \ldots w_n$ of W one has $\det(b(v_i, w_j)) \neq 0$.
   (b) Produce such a bilinear form in the case where $V = \Lambda^p U$ and $W = \Lambda^{k-p} U$ when U is a k-dimensional vector space over F.
   (c) To what extent may one generalize the two preceding items to correct statements when F is a commutative ring, V, W free F-modules of rank n, and U a free F-module of rank k?