Math 520B
Written Assignment No. 1

due Monday, October 10, 2005

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use. You may not seek help from others.

In this problem set the notation $\hat{\mathbf{Z}}_m$ for a positive integer m denotes the ring given by

$$\hat{\mathbf{Z}}_m \;=\; \text{proj lim}_{n\to\infty}\, \mathbf{Z}/m^n\mathbf{Z} \quad .$$

1. Let A be a commutative ring, M an A-module, and G a group that acts on M by A-linear automorphisms. Show that there is one and only one way to endow M with the structure of module over the group ring AG (of G with coefficients in A) that is compatible with the given A-module structure on M and such that the AG-module scalar multiplication of any element $x \in M$ by $g \in AG$ (i.e., by $\delta(g)$ where δ is the canonical group homomorphism $G \to AG^*$) has the same result as does applying g to x via the given group action of G on M.

2. Let R be a ring, M and N (left) R-modules, $P = M \times N$ the product of M and N, $M \stackrel{i}{\to} P$ the map $m \mapsto (m, 0)$, and $N \stackrel{j}{\to} P$ the map $n \mapsto (0, n)$. Show that the pair (i, j) has the property that for any R-module X and any pair (f, g) of R-linear maps with $M \stackrel{f}{\to} X$ and $N \stackrel{g}{\to} X$ there is one and only one R-linear map $P \stackrel{h}{\to} X$ such that $h \circ i = f$ and $h \circ j = g$.

3. A (left) module M on a commutative ring R always gives rise to a bi-R-module with the property that for all $r \in R$ and $m \in M$ one has $r \cdot m = m \cdot r$. Let R be the polynomial ring $\mathbf{F}_2[t]$ where $\mathbf{F}_2$ is the field of two elements, and let M be the two-dimensional Cartesian space $\mathbf{F}_2 \times \mathbf{F}_2$. Give an example of an R-bi-module structure on M for which the right multiplication by a scalar is not equal to the left multiplication.

4. Regarding the rings $\hat{\mathbf{Z}}_m$ prove the following:

   (a) If $\gcd(m, n) = 1$, then
   $$\hat{\mathbf{Z}}_{mn} \cong \hat{\mathbf{Z}}_m \times \hat{\mathbf{Z}}_n \quad .$$

   (b) For each integer $n \geq 1$
   $$\hat{\mathbf{Z}}_{m^n} \cong \hat{\mathbf{Z}}_m \quad .$$

5. For a prime p let $\mathbf{Q}_p$ denote the field of fractions of the ring $\hat{\mathbf{Z}}_p$. Prove the following statements:

   (a) The set of non-units in $\hat{\mathbf{Z}}_p$, i.e., $\hat{\mathbf{Z}}_p - \left(\hat{\mathbf{Z}}_p\right)^*$, is an additive subgroup of $\hat{\mathbf{Z}}_p$.

   (b) For any $x \neq 0$ in $\mathbf{Q}_p$ at least one of the two elements $\{x, 1/x\}$ must be in $\hat{\mathbf{Z}}_p$.