# Math 520B

# Written Assignment No. 1

### due Friday, September 16, 2005

**Directions.** It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use. You may not seek help from others.

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1. Let $K$ be the splitting field over $\mathbf{Q}$ of the polynomial $t^8 - 1$, and let $L = K(\sqrt[8]{2})$.
   (a) Show that $L$ is the splitting field of $t^8 - 2$ over $\mathbf{Q}$.
   (b) Find the Galois group of $K$ over $\mathbf{Q}$.
   (c) Find the Galois group of $L$ over $K$.
   (d) Find the Galois group of $L$ over $\mathbf{Q}$.

2. Let $K$ be a field, $L$ a (finite) Galois extension of $K$, $G$ the Galois group, and $R = KG$ the group ring of $G$ with coefficients in $K$. Observe that $L$ and $R$ have the same dimension as vector spaces over $K$.
   (a) Show that there is an obvious $R$-module structure on $L$.
   (b) Formulate in terms of the study of $L$ as an extension of $K$ without reference to the notion of group ring what it means for $L$ to be isomorphic as an $R$-module to $R$.

3. How small can a non-commutative ring be? Prove the following:
   (a) Every ring $R$ with $|R| \leq 7$ is commutative.
   (b) If $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ is the field of 2 elements, there is a subring of the $2 \times 2$ matrix ring $\mathrm{M}_2(\mathbf{F}_2)$ over $\mathbf{F}_2$ that is non-commutative and has 8 elements.

4. In Hungerford's text at p. 227, Defn. 7.1, an "algebra" $A$ over a commutative ring $K$ is defined to be a ring $A$, not necessarily commutative, that is also a $K$-module where it is required that the $K$-module structure $K \times A \xrightarrow{\cdot} A$ is related to the ring multiplication of $A$ by the formulae
   $$(k \cdot a_1)a_2 \;=\; k \cdot (a_1 a_2) \;=\; a_1(k \cdot a_2) \quad .$$
   Prove that there is a ring homomorphism $K \xrightarrow{\varphi} A$ such that for all $k \in K$ and $a \in A$ one has
   $$k \cdot a \;=\; \varphi(k)a \;=\; a\varphi(k) \quad .$$
   (Be sure to begin by defining $\varphi$.)

5. Let $F$ be any field, and let $L = F(t_1, \ldots, t_n)$ be the field of rational functions in $n$ variables over $F$. Find a subfield $K$ of $L$ with the property that $L$ is a cyclic extension of $K$ of degree $n$.