Math 520B Written Assignment No. 5

due Wednesday, December 8, 2004

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use.

You may not seek help from others.

1. Let R be a commutative ring, P a prime ideal in R, and R_P the localization of R at P. Show that the fraction field of the domain R/P is isomorphic to $R_P/P \cdot R_P$.

2. For a prime $p > 1$ in $\mathbf{Z}$ let $\mathbf{Q}_p$ denote the field that is the completion of $\mathbf{Q}$ with respect to its p-adic valuation.

   (a) Show that -1 is not the square of any element of $\mathbf{Q}_7$.

   (b) Show that there is an element c in the valuation ring of $\mathbf{Q}_5$ such that $c^2 = -1$.

3. Let $\mathcal{O}$ be the ring of power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

   with complex coefficients $\{c_k\}$ in the variable z each of which converges in some disk in $\mathbf{C}$ with center 0.

   (a) Explain how to construct a valuation on the fraction field of $\mathcal{O}$ using the "order of vanishing" at 0 of a power series f.

   (b) What concrete ring is the valuation ring in the completion of the fraction field of $\mathcal{O}$ with respect to this valuation?

4. For a given field K determine up to equivalence all valuations of the field $K(t)$ of "rational functions" with coefficients in K that are trivial on K (embedded as the subfield of constants in $K(t)$).

5. If F is a field and V, W are finite-dimensional vector spaces over F, prove that there is a *natural* isomorphism
$$V^* \otimes_F W \longrightarrow \mathrm{Hom}_F(V, W)$$
   where $V^* = \mathrm{Hom}_F(V, F)$ is the vector space over F dual to V.