Math 520B Written Assignment No. 2

due Monday, October 4, 2004

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use.

You may not seek help from others.

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1. Prove that if $R \neq (0)$ is a finite ring with the property that there is no relation $xy = 0$ with $x \neq 0$ and $y \neq 0$ in R, then R is a division ring.

2. How small can a non-commutative ring be? Prove the following:

   (a) Every ring R with $|R| \leq 7$ is commutative.

   (b) If $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ is the field of 2 elements, there is a subring of the 2×2 matrix ring $\mathrm{M}_2(\mathbf{F}_2)$ over $\mathbf{F}_2$ that is non-commutative and has 8 elements.

3. If $\mathbf{F}_2$ is the field of 2 elements, how many ring homomorphisms are there from the polynomial $\mathbf{Z}[t]$ to the 2×2 matrix ring $\mathrm{M}_2(\mathbf{F}_2)$?

4. Let R be a ring, M and N (left) R-modules, $P = M \times N$ the product of M and N, $M \xrightarrow{i} P$ the map $m \mapsto (m, 0)$, and $N \xrightarrow{j} P$ the map $n \mapsto (0, n)$. Show that the pair (i, j) has the property that for any R-module X and any pair (f, g) of R-linear maps with $M \xrightarrow{f} X$ and $N \xrightarrow{g} X$ there is one and only one R-linear map $P \xrightarrow{h} X$ such that $h \circ i = f$ and $h \circ j = g$.

5. In Hungerford's text at p. 227, Defn. 7.1, an "algebra" A over a commutative ring K is defined to be a ring A, not necessarily commutative, that is also a K-module where it is required that the K-module structure $K \times A \xrightarrow{\cdot} A$ is related to the ring multiplication of A by the formulae
$$(k \cdot a_1)a_2 = k \cdot (a_1 a_2) = a_1(k \cdot a_2) \quad .$$
   Prove that there is a ring homomorphism $K \xrightarrow{\varphi} A$ such that for all $k \in K$ and $a \in A$ one has
$$k \cdot a = \varphi(k)a = a\varphi(k) \quad .$$
   (Be sure to begin by defining φ.)

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