Math 520A

Written Assignment No. 5

due Monday, May 7, 2007

Directions. This assignment should be typeset. You must explain the reasoning underlying your answers. If you make use of a reference other than class notes, you must properly cite its use.

You may not seek help from others on this assignment.

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1. Write your own proofs of the following propositions:

   (a) Every polynomial of degree 1 with coefficients in a field is irreducible.

   (b) A field F admits no non-trivial algebraic extension if and only if every irreducible polynomial with coefficients in F has degree 1.

2. If F is a field, a polynomial $f(t) \in F[t]$ with coefficients in F determines a "polynomial function" $\varphi(f)$ from F to itself that is defined by

$$(\varphi(f))(a) \;=\; f(a) \text{ for } a \in F \quad .$$

   If A denotes the F-algebra of all functions $F \to F$, φ is an F-algebra homomorphism $F[t] \to A$. Show the following:

   (a) φ is injective if F is an infinite field.

   (b) φ is not injective if F is a finite field.

   (c) φ is not surjective if F is an infinite field.

   (d) φ is surjective if F is a finite field.

3. A *primitive element* for a field extension E/F is an element $\theta \in E$ such that $E = F(\theta)$. Find primitive elements for E over $\mathbf{Q}$ in the following cases:

   (a) E is the splitting field over $\mathbf{Q}$ of $t^{12} - 1$.

   (b) $E = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

4. More on the polynomial $t^4 + 1$:

   (a) Explain why $t^4 + 1$ is irreducible in $\mathbf{Q}[t]$.

   (b) Show that $t^4 + 1$ is **not** irreducible over $\mathbf{Z}/p\mathbf{Z}$ for every prime p.

   (c) Find the group of $\mathbf{Q}$-algebra automorphisms of the field

$$\mathbf{Q}[t]/(t^4 + 1)\mathbf{Q}[t] \quad .$$

5. For each of the following irreducible polynomials with coefficients in $\mathbf{Q}$ determine the Galois group over $\mathbf{Q}$ of its splitting field:

   (a) $t^3 - 4t + 2$.

   (b) $t^3 - 3t - 1$.

   (c) $t^4 - 2t^2 - 1$.

   (d) $t^4 - 4t^2 + 2$.

   (e) $t^4 - 10t^2 + 1$.