Math 520A Written Assignment No. 4

due Monday, April 23, 2007

Directions. This assignment should be typeset. You must explain the reasoning underlying your answers. If you make use of a reference other than class notes, you must properly cite its use.

You may not seek help from others on this assignment.

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1. Decompose the polynomial $t^{12} - 1 \in F[t]$ as the product of irreducible polynomials when F is the field
   (a) $\mathbf{Q}$.
   (b) $\mathbf{Z}/5\mathbf{Z}$.
2. Let A denote the ring
   $$\mathbf{R}[t]/(t^4+1)\mathbf{R}[t]$$
   and π the quotient homomorphism
   $$\pi : \mathbf{R} \to \ \mathbf{R}[t]/(t^4+1)\mathbf{R}[t] \ ;$$
   observe that A is an $\mathbf{R}$-algebra via π.
   (a) Determine the group of $\mathbf{R}$-algebra automorphisms of A.
   (b) Assuming as known the fact (a consequence of the "fundamental theorem of algebra") that, up to $\mathbf{R}$-algebra isomorphism, the only non-trivial finite extension of the field $\mathbf{R}$ is $\mathbf{C}$, find all subfields of A that contain $\pi(R)$.
3. Recall that the multiplicative group of a finite field must be cyclic. For the irreducible polynomial $p(t) \in F[t]$ find a polynomial in $F[t]$ of degree 1 whose congruence class mod $p(t)$ determines a generator for the multiplicative group of the finite field $F[t]/(p(t))F[t]$ when
   (a) $F = \mathbf{Z}/2\mathbf{Z}, \quad p(t) = t^4 + t + 1.$
   (b) $F = \mathbf{Z}/3\mathbf{Z}, \quad p(t) = t^2 + 1.$
   (c) $F = \mathbf{Z}/3\mathbf{Z}, \quad p(t) = t^3 - t - 1.$
   (d) $F = \mathbf{Z}/2\mathbf{Z}, \quad p(t) = t^5 + t^2 + 1.$
4. Find a monic polynomial $q(t)$ of degree 4 with integer coefficients having
   $$\begin{aligned} \alpha &= \sqrt{2} + \sqrt{3} + \sqrt{6} \\ \beta &= -\sqrt{2} + \sqrt{3} - \sqrt{6} \\ \gamma &= \sqrt{2} - \sqrt{3} - \sqrt{6} \\ \delta &= -\sqrt{2} - \sqrt{3} + \sqrt{6} \end{aligned}$$
   as real roots. Explain why $q(t)$ must be irreducible in $\mathbf{Q}[t]$.
5. For each of the following monic polynomials p of degree 4 with coefficients in $\mathbf{Q}$ determine the extension degree over $\mathbf{Q}$ of the smallest subfield of $\mathbf{C}$ in which all complex roots of p lie:
   (a) $t^4 - 10t^3 + 35t^2 - 50t + 24.$
   (b) $t^4 + 2.$
   (c) $t^4 - 2t^2 - 1.$
   (d) $t^4 + t - 1.$
   (e) $t^4 - 4t^2 + 2.$